International Journal of Basic Sciences and Applied Computing (IJBSAC)
ISSN: 2394-367X (Online), Volume-3 Issue-4, August 2021

On the Solutions of Diophantine Equation $(M_p - 2)^x + (M_p + 2)^y = z^2$ where M_p is Mersenne Prime

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Abstract: ***The Diophantine equation has been studied by many researchers in number theory because it helps in solving variety of complicated puzzle problems. From several studies, many interesting proofs have been found. In this paper, the researcher has examined the solutions of Diophantine equation $(M_p - 2)^x + (M_p + 2)^y = z^2$ where M_p is a Mersenne Prime and p is an odd prime whereas x, y and z are nonnegative integers. It was found that this Diophantine equation has no solution.***

Keywords: Diophantine equations, exponential equations.

I. INTRODUCTION

The Diophantine equation is an equation in which only integer's solutions are allowed. The Diophantine equations is a major part of number theory. In current years, many researchers have tried to find the solutions to the Diophantine equation of the form; $p^x + q^y = z^2$, in which p and q are primes, where x, y and z are non-negative integers.

In the year 2013, S. Chotchaisthit [2] found that the Diophantine equation $p^x + (p+1)^y = z^2$ has no solution except for p = 3, 7 in which p is Mersenne prime and x, y, and z are non-negative integers.

Additionally in 2018, S.Kumar, S.Gupta and H.Kishan [5] showed that the non-linear Diophantine equation $p^x + (p+6)^y = z^2$, when p and p+6 are both primes with p = 6n+1 and n is a natural number and has no solution in non-negative integer x, y and z.

In addition the above exemples, Fernando [4] also investigated the equation and proved that the Diophantine equation $p^x + (p+8)^y = z^2$ where $p > 3$ and p+8 are primes has no solution.

In 2019, A. Sugandha [8], A. Prabowo, A. Tripena examined solutions to Non-Linear Diophantine Equation $p^x + (p+5)^y = z^2$ when p is Mersenne Prime.

In 2020, Burshtein [1] showed that the Diophantine $p^x + (p+5)^y = z^2$ has no solution in positive integer x, y and z in which p is prime and $p + 5 = 2^{2u}$ and proved that the Diophantine $p^x + q^y = z^3$ has no solution when $p \geq 2$, p and q are primes where $x \geq 1, y \leq 2$ are integers.

Apart from what stated earlier, in 2021, Moonchaisook [7] examined the non-linear Diophantine equation and proved that $p^x + (p+4^n)^y = z^2$ has no solution where x, y and z are non-negative integer.

In 2021, the Diophantine $M_p^x + (M_q + 1)^y = z^2$ has solutions $(M_p, 7, 0, 1, 3), (3, M_q, 1, 0, 2)$ and $\left(M_p, M_q, 2, \frac{p+2}{q}, 2^p + 1\right)$ was proved by W.S. Gayo, et.al [6]. Additionally, in the same year, A. Pakapongpun and R. Dochan [3] studied the Diophantine equation $p^x + (p+20)^y = z^2$ and proved that this equation has no solution where p and p+20 are primes.

Because of this open problem, the author is therefore interested in studying the Diophantine equation $(M_p - 2)^x + (M_q + 2)^y = z^2$ has no solution where x, y and z are non-negative integer and M_p is a Mersenne prime.

II. PRELIMINARIES

Definition. We can call every prime number written in the form of $2^p - 1$ a Mersenne prime which can be replaced by M_p is which p is a prime number.
That is $M_p = 2^p - 1$

Lemma 2.1 If z is an even number. Then z^2 is an even number and $4|z^2$

Proof: Suppose z is an even number of the form z = 2m where m is whole number. So $\boldsymbol{z^2} = \boldsymbol{4m^2}$ is an even number $\rightarrow \quad 4|z^2$

Lemma 2.2 If p is an odd prime. Then $M_p \equiv 3(mod\ 4)$ where $M_p = 2^p - 1$

Proof: Since $\mathrm{M_p} = 2^p - 1$
$\rightarrow \mathrm{M_p} = 2^2(2^{p-2}) - 1 \equiv -1(mod 4)$
$\rightarrow \mathrm{M_p} \equiv 3(\mathrm{mod}\ 4)$

Lemma 2.3 The Diophantine equation $(\mathrm{M_p} - 2)^x + 1 = z^2$ has no solution in non-negative integer x, z where $\mathrm{M_p}$ is a Mersenne prime and p is an odd prime.

Manuscript received on August 10, 2021.
Revised Manuscript received on August 22, 2021.
Manuscript published on August 30, 2021.
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Proof: Suppose $(M_p-2)^x+1=z^2$
where $M_p=2^p-1$ and p is an odd prime.
We consider in 2 cases as follows:
Case 1. Suppose x = 0. Thus $z^2=2$ which is impossible.
Case 2. If x ≥ 1 and $M_p=2^p-1$.
Since $(M_p-2)^x+1=z^2$.
Then $(M_p-2)^x=(z+1)(z-1)$
Let α and β are non-negative integer. Suppose
$(M_p-2)^\alpha=z-1$ and $(M_p-2)^\beta=z+1$
with $\alpha<\beta, \alpha+\beta=x$
$\rightarrow \quad (M_p-2)^\alpha[(M_p-2)^{\beta-\alpha}-1]=2$
This implies that $\alpha=0$, and $\beta=1$
So $M_p=5 \rightarrow 2^p-1=5$
Thus $2^p=6$ it is obtained $p=\log_2 6$ which is not odd prime. Hence, the Diophantine equation $(M_p-2)^x+1=z^2$ has no solution.

Lemma 2.4 If $M_p$is Mersenne prime and p is an odd prime. Then the Diophantine $1+(M_p+2)^y=z^2$ where y and z are non-negative integer has no solution.
Proof: Let $1+(M_p+2)^y=z^2$
We consider in 2 cases as follows:
Case 1. Suppose that y = 0 Thus $z^2=2$ which is impossible.
Case 2. Suppose that y ≥ 1
We have $(M_p+2)^y=z^2-1=(z-1)(z+1)$.Thus, we can fine two non-negative integers α and β.
Let $(M_p+2)^\alpha=z-1$ and $(M_p+2)^\beta=z+1$
with $\alpha<\beta, \alpha+\beta=y$
$\rightarrow \quad (M_p+2)^\alpha[(M_p+2)^{\beta-\alpha}-1]=2$
which implies that $\alpha=0$, and $\beta=1$
$\rightarrow \quad (M_p+2)^\beta-1=2 \rightarrow M_p+2=3$
$\rightarrow \quad 2^p=2$ which is impossible. Since p is an odd prime. Hence, the Diophantine equation
$1+(M_p+2)^y=z^2$ has no solution.

III. MAIN THEOREM

Theorem 3.1. If M_p is a Mersenne prime and p is an odd prime then the Diophantine equation
$(M_p-2)^x+(M_p+2)^y=z^2$ has no solution in non-negative integer x, y and z

Proof: Let $(M_p-2)^x+(M_p+2)^y=z^2$.
We consider in 3 cases as follows:

Case 1. If x = 0, then $1+(M_p+2)^y=z^2$.
has no solution by lemma 2.4.
Case 2. If y = 0, then $(M_p-2)^x+1=z^2$
has no solution by Lemma 2.3.
Case 3. If x ≥ 1 and y ≥ 1, then both $(M_p-2)^x$and $(M_p+2)^y$are odd number thus $(M_2-2)^x+(M_2+2)^y=z^2$ is an even number.
Now, we have $M_p=2^p-1$
$\rightarrow \quad 2^2(2^{p-2})-3 \equiv 1 (mod\ 4)$
$\rightarrow \quad 2^2(2^{p-2})+1 \equiv 1 (mod\ 4)$
Thus $(M_p-2)^x+(M_p+2)^y \equiv 2\ (mod\ 4)$ (1)
From equation (1), $z^2 \equiv 2$ (mod 4) is contradiction lemma 2.1. For this reason, $(M_p-2)^x+(M_p+2)^y=z^2$ has no solution.

Corollary 3.1.1. If M_p is a Mersenne prime and p is an odd prime so the Diophantine equation $(M_p-2)^x+(M_p+2)^y=u^{2n}$ when x, y and z are non-negative integers has no solution.
Proof: Let $u^n=z$.
Then $(M_p-2)^x+(M_p+2)^y=u^{2n}=z^2$ has no solution by Theorem 3.1.

Corollary 3.1.2. If M_p is a Mersenne prime and p, q are distinct primes of the form $p=4t+1$or $p=4t+3$ and $q=4r+1$, the Diophantine equation $(M_p-2)^x+q^y=z^2$ when x, y and z are non-negative integers and r and t are positive integers has no solution.

Proof. Suppose the Diophantine equation
$$(M_p-2)^x+q^y=z^2.$$

We consider 2 cases as follows:
Case 1. If $p=4t+1, q=4r+1$ then
$(M_p-2)^x+q^y=z^2 \equiv 2$ (mod 4). However, this implies that $4 \nmid z^2$, contradicts the Lemma 2.1.
For this reason, $(M_p-2)^x+q^y=z^2$ has no solution.

Case 2. If $p=4t+3, q=4r+1$. then
$(M_p-2)^x+q^y=z^2 \equiv 3(mod\ 4)$. However, this implies that $4 \nmid z^2$ contradicts the Lemma 2.1 thus $(M_p-2)^x+q^y=z^2$. has no solution.

Corollary 3.1.3. If M_p is a Mersenne prime and $M_p=2^p-1$ which p and q are primes, the Diophantine equation $(M_p+2)^x+q^y=z^2$ has no solution.

Proof. Suppose the Diophantine equation
$(M_p+2)^x+q^y=z^2$

Case 1. If $p=4t+1, q=4r+1$ then
$(M_p+2)^x+q^y=z^2 \equiv 2\ (mod\ 4)$. However, this implies that $4 \nmid z^2$ contradicts the Lemma 2.1
thus $(M_p+2)^x+q^y=z^2$ has no solution.
Case 2. If $p=4t+3, q=4r+1$,
$(M_p+2)^x+q^y=z^2 \equiv 3(mod\ 4)$. However, this implies that $4 \nmid z^2$ contradicts the Lemma 2.1.
thus $(M_p+2)^x+q^y=z^2$ has no solution.

IV. CONCLUSION

From this investigation, it was found that the Diophantine equation $(M_p - 2)^x + (M_p + 2)^y = z^2$ and showed that $(M_p - 2)^x + (M_p + 2)^y = z^2$has no solution in which x, y and z are non-negative and m, n are natural numbers. There are infinitely many the Diophantine equation that have no solution, which need to be proved.

ACKNOWLEDGEMENTS

The author is very appreciated with the useful remarks and guidance from the auditorial board in completing the paper.

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