On the Invariance Property for S-Flows in Topological Dynamics System

Abdulkafi A. Al-Rafaei, Amin Saif

Abstract: In this paper, we start by giving an equivalence relation on a topological space X which correspond, under the action of a topological monoid S, to the S-invariant control sets for control systems. Then we give some results about the S-invariant classes for this relation. The conditions for the existence and uniqueness of relative S-invariant classes will be given.

Keywords: Topological monoid; S-flow; S-phase flow. AMS classification: 06F05, 76D55.

I. INTRODUCTION

The invariance theory is one of the principal concepts in the topological dynamics [2, 4]. Colonius and Kliemann in [3] introduced the concept of a control system which is acted by a topological monoid S. Throughout this paper, we will define an equivalence relation on a topological space X which correspond, under the action of a topological monoid S, to the S-invariant control sets. 

II. PRELIMINARIES

Let S be a monoid with the identity element e and also a topological space. S will be called a topological monoid if the multiplication operation of S : (s, t) → st is a continuous mapping from S × S to S.

Definition 2.2. [6] Let S be a topological monoid and X be a topological space. We say that S acts on X as a transformation semigroup if there is a continuous map a : S × X → X between the product space S × X and X satisfying a(st, x) = a(s, a(t, x)) for all s, t ∈ S, x ∈ X.

We further require that a(e, x) = x for all x ∈ X. The triple (S, X, a) is called an S-flow.

Lemma 2.3. [Zorn’s Lemma] [5] If each chain in a partially ordered set has an upper bound, then there is a maximal element of the set.

3. S-invariant classes

Let (S, X, a) be an S-flow. From the action on X we can define the relation ∼ on X by

x ∼ y if x ∈ Oa(y) and y ∈ Oa(x), x, y ∈ X.

It’s clear that ∼ is an equivalence relation and [X] will denote the set of all equivalence classes induced by ∼ on X. We observe that [x] ⊆ Oa(x) for all x ∈ X, and if y ∈ Oa(x), then Oa(y) ⊆ Oa(x) for all x, y ∈ X.

Theorem 3.1. Let (S, X, a) be an S-phase flow. A class [x] ∈ [X] with intX([x]) ≠ ∅ is a control set for S on X.

Proof. It’s clear that [x] ⊆ Oa(x) ⊆ Oa(y) ⊆ cl(Oa(y)) for all y ∈ [x]. Suppose C be a subset of X satisfying the property

C ⊆ cl(Oa(z)) for all z ∈ C and [x] ⊆ C.
On the Invariance Property for $S$−Flows in Topological Dynamics System

Let $w \in C$. Then $w \in \text{cl}(O_a(z))$ for all $z \in C$. Since $S$ is a compact space, $X$ is a Hausdorff space and by the continuity of the action $a$, then the orbit $O_a(x)$ is a closed subset of $X$ for all $x \in X$ (i.e., $\text{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Then $w \in O_a(z)$ for all $z \in C$. Since $x \in C$, then $w \in O_a(x)$. On the other hand, since $x \in [x] \subset C \supseteq O_a(w)$, then $w \in [x]$. Hence $C = [x]$.

In the following lemma, we give the necessary and sufficient conditions for the equivalence classes to be $S$−invariant classes.

**Lemma 3.2.** Let $(S, X, a)$ be an $S$−flow. A class $[x] \in [X]$ is an $S$−invariant class if and only if $[x] = O_a(x)$.

Proof. Suppose $[x] \in [X]$ is an $S$−invariant and let $y \in O_a(x)$, then $y = s\bar{a}x$ for some $s \in S$. Since $x \in [x]$, then $y \in S([x]) \subset [x]$. Hence $O_a(x) \subset [x]$, and we have $[x] \subset O_a(x)$.

Therefore $[x]$ is $S$−invariant class.

**Theorem 3.3.** Let $(S, X, a)$ be an $S$−phase flow. Then for all $x \in X$, there is a unique $S$−invariant class $[y] \subset O_a(x)$.

Proof. For $x \in X$, consider the family of subsets

$$Ex = \{z : O_a(z) \subset O_a(x)\}.$$ 

We can define the relation $\leq$ on Ex by

$$x_1 \leq x_2 \text{ if } O_a(x_2) \subset O_a(x_1) \text{ for } x_1, x_2 \in E_x.$$ 

It’s clear that the family $E_x$ with $\leq$ is a partially ordered set. Let $\{z_i : i \in \Lambda\}$ be a linearly ordered subset of $E_x$, where $\Lambda$ is an index set. Since $S$ is a compact space, $X$ is a Hausdorff space and by the continuity of the action $a$, then the orbit $O_a(x)$ is a compact closed subset of $X$ for all $x \in X$. Hence we have a chain $\{O_a(z_i) : i \in \Lambda\}$ of closed subsets of a compact $O_a(x)$. Hence the intersection

$$\bigcap_{i \in \Lambda} O_a(z_i) \neq \emptyset.$$ 

Take $r \in O_a(z_i)$ for all $i \in \Lambda$. Then $O_a(r) \subset O_a(z_i)$ for all $i \in \Lambda$, imply that $O_a(r)$ is a lower bound of the chain $\{O_a(z_i) : i \in \Lambda\}$ (i.e., $r$ is an upper bound of the linearly order subset $\{z_i : i \in \Lambda\}$ of $E_x$). Hence by Zorn’s lemma implies that the family $E_x$ has a maximal element, say $y$. Then $[y] \subset O_a(y) \subset O_a(x)$.

Now, we show that $[y]$ is an $S$−invariant. Let $z \in O_a(y)$, then $z \in O_a(z) \subset O_a(x)$ and $y \leq z$, but by the maximality of $y$, we get that $z \leq y$, this implies $y \in O_a(z)$. Hence $z \in [y]$ (i.e., $O_a(y) \subset [y]$) and we have that $[y] \subset O_a(y)$. Then by Lemma 3.2, $[y]$ is an $S$−invariant class. Now, let $[\alpha] \neq [y]$ be an $S$−invariant class such that $[\alpha] \subset O_a(x)$. Then $O_a(\alpha) \subset O_a(x)$. Hence $\alpha \in E_x$. By the maximality of $y$, we get that $\alpha \leq y$, this implies that $[y] = O_a(y) \subset O_a(\alpha) = [\alpha]$.

Hence $[y] = [\alpha]$, this means that $[y]$ is a unique.

**Theorem 3.4.** Let $(S, X, a)$ be an $S$−phase flow. $[x] \in [X]$ has no-return condition for all $x \in X$.

Proof. Since $S$ is a compact space, $X$ is a Hausdorff space and by the continuity of the action $a$, then the orbit $O_a(x)$ is a compact closed subset of $X$ for all $x \in X$ (i.e., $\text{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Let $z \in O_a(y)$ for some $y \in [x]$ and $O_a(z) \cap [x] \neq \emptyset$. Take $w \in O_a(y)$ and $w \in [x]$. Hence $x \in O_a(w) \subset O_a(y) \subset O_a(w)$. On the other hand, $z \in O_a(y)$ for some $y \in [x]$, we have $z \in O_a(z) \subset O_a(y) \subset O_a(x)$. Hence $z \in [x]$.

The next theorem clears that if $M$ has the no-return condition, then any a class $[x]$ is entirely contained in $M$ or $M^c$. Also $M$ is an $S$−invariant if $[x]$ is an $S$−invariant class for all $x \in M$.

**Theorem 3.5.** Let $(S, X, a)$ be an $S$−phase flow and $M$ be a subset of $X$ has no-return condition. $M$ is an $S$−invariant set if $[x] \in [X]$ is an $S$−invariant class for all $x \in M$.

Proof. It’s clear that $M \subset \bigcup_{x \in M} [x]$ because $x \in [x]$. Since $S$ is a compact space, $X$ is a Hausdorff space and by the continuity of the action $a$, then the orbit $O_a(x)$ is a compact closed subset of $X$ for all $x \in X$ (i.e., $\text{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Let $y \in \bigcup_{x \in M} [x]$, then $y \in [x]$ for some $x \in M$. Hence $[y] = [y]$ (i.e., $x \in O_a(y)$ and $y \in O_a(x)$). Since $x \in M$, then $O_a(y) \cap M \neq \emptyset$. By the no-return condition we have that $y \in M$. Hence

$$M = \bigcup_{x \in M} [x].$$ 

Now, we show that $M$ is an $S$−invariant set. Let $y \in S(M)$. Then $y = s\bar{a}x$ for some $x \in M$. Hence $y \in O_a(x)$. Since $[x]$ is an $S$−invariant class then by Lemma 3.2, $[x] = O_a(x)$ and by Equation (1), we get that $y \in [x] \subset M$. Hence $M$ is an $S$−invariant.
REFERENCES

3. F. Colonius, W. Kliemann, Linear control semigroups acting on projective systems,

AUTHORS PROFILE

First Author
Name: Abdulkafi Abdullahattah Saleh Al-Refaei
Date of Birth: November 019, 1974
Place of Birth: IBB, Yemen
Educational skills:
- PhD school of Mathematics, Faculty Science and Technology National University of Malaysia (Universiti Kebangsaan Malaysia UKM) 2007.
- M. Sc. Math.: Department of Mathematic, Moustansaeria University, Iraq, 2002.
- B. Sc. Math.: Department Mathematic, TAIZ University, Yemen, 1996

Second Author
Name: Ameen hamood Saif Al-sunawi
Taiz University Faculty of education Department of mathematics
Assistant of Professor

SCIENTIFIC TITLES
- Assistant professor in 26-11-2008 Faculty of Applied sciences Thamar University, Yemen.
- Associate Professor in 26-10-2014 Faculty of Applied sciences Thamar University, Yemen.

- Vice President for Academic Affairs, Damar University, 2017-2018
- Member of the Academic Council September 2015 until 2018.
- Head of Library, Printing, Publishing and Translation Department, Damar University, January 2015.
- Vice Dean for Graduate Studies in the Faculty of Applied sciences, Thamar University, Yemen, since Die 2012.
- Head of Mathematic in the Faculty of Applied sciences, Thamar University, Yemen, since Nov 2010 until Dicember 2012.
- Served as Lecturer in the Faculty of Applied sciences, Thamar University, Yemen, since Jan 2008 until present.
- Served as Lecturer in the Faculty of Education and Science, Sana’a University, Arhab 2007.