On the Invariance Property for S−Flows in Ttopological Dynamics System

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Abstract: In this paper, we start by giving an equivalence relation on a topological space X which correspond, under the action of a topological monoid S, to the S−invariant control sets for control systems. Then we give some results about the S−invariant classes for this relation. The conditions for the existence and uniqueness of relative S−invariant classes will be given.

Keywords: Topological monoid; S−flow; S−phase flow. AMS classification: 06F05, 76D55.

I. INTRODUCTION

The invariance theory is one of the principal concepts in the topological dynamics system [2, 4]. Colonius and Kliemann in [3] introduced the concept of a control set which is relatively invariant with respect to a subset of the phase space of the control system. From a more general point of view, the theory of control sets for semigroup actions was developed by San Martin and Tonelli in [6].

In this paper, we will define an equivalence relation on a topological space which is acted by topological monoid S as a transformation semigroup. Then we study the S−invariant classes for this relation in X, in particular, the conditions for the existence and uniqueness of S−invariant classes will be given.

II. PRELIMINARIES

Throughout this paper, *cl(A)* will denote the closure set of a set *A*, *int(A)* the interior set of A and all topological spaces involved Hausdorff.

Definition 2.1. [4] Let S be a monoid with the identity element e and also a topological space. S will be called a *topological monoid* if the multiplication operation of $S : (s, t) \quad st$ is continuous mapping from $S \times S$ to S.

Definition 2.2. [6] Let S be a topological monoid and X be a topological space. We say that S *acts on* X *as a transformation semigroup* if there is a continuous map $a : S\times X \quad X$ between the product space S × X and X satisfying

$a(st,x) = a(s,a(t,x))$ for all $s,t \in S, x \in X$;

we further require that $a(e, x) = x$ for all $x \in X$. The triple (S , X , a) is called an S−*flow*;

$s\bar{a}x$ will denote $a(s, x)$. An S−flow (S , X , a) is called S−*phase flow* if S is a compact space.

The *orbit* of $x \in X$ under S is the set $O_a(x) = \{s\bar{a}x \ : \ s \in S\}$. For a subset M of X, $S(M)$ denotes the set $\{s\bar{a}m \ : \ s \in S, \ m \in M\}$. And a subset M is called an S−*invariant* set if $M \neq \emptyset$ and $S(M) \subset M$. A *control* set for S on X is a subset C of X which satisfies:

1. $\mathrm{int}(C) \neq \emptyset$;
2. for all $x \in C$, $C \subset cl(O_a(x))$;
3. C is a maximal with these properties.

We say that a subset $M \subset X$ satisfy the no-return condition if $y \in \mathrm{cl}(Oa(x))$ for some $x \in M$ and $\mathrm{cl}(Oa(y)) \cap M \neq \emptyset$, then $y \in M$.

Lemma 2.3. [Zorn's Lemma][5] If each chain in a partially ordered set has an upper bound, then there is a maximal element of the set.

3 S−invariant classes

Let (S , X , a) be an S−flow . From the action on X we can define the relation $\sim$ on X by

$$x \sim y \quad \text{if} \quad x \in O_a(y) \quad \text{and} \quad y \in O_a(x), \quad x, y \in X .$$

It's clear that $\sim$ is an equivalence relation and [X] will denote the set of all equivalence classes induced by $\sim$ on X. We observe that $[x] \subset Oa(x)$ for all $x \in X$, and if $y \in O_a(x)$, then $O_a(y) \subset O_a(x)$ for all $x, y \in X$.

The following theorem shows that ant equivalence class with nonempty interior set is a control set for S on X.

Theorem 3.1. *Let* (S,X, a) *be an* S−*phase flow. A class* $[x] \in [X]$ *with* $intX([x]) \neq \emptyset$ *is a control set for* S *on* X.

Proof. It's clear that $[x] \subset O_a(x) \subset O_a(y) \subset cl(O_a(y))$ for all $y \in [x]$. Suppose C be a subset of X satisfying the property

$C \subset cl(O_a(z))$ for all $z \in C$ and $[x] \subset C$.

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Let $w \in C$. Then $w \in cl(O_a(z))$ for all $z \in C$. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $Oa(x)$ is a closed subset of X for all $x \in X$ (i.e., $cl(O_a(x)) = O_a(x)$ for all $x \in X$). Then $w \in O_a(z)$ for all $z \in C$. Since $x \in C$, then $w \in O_a(x)$. On the other hand, since $x \in [x] \subset C \subset O_a(w)$, then $w \in [x]$. Hence $C=[x]$.

In the following lemma, we give the necessary and sufficient conditions for the equivalence classes to be S−invariant classes.

Lemma 3.2. *Let (S , X , a) be an S−flow. A class* $[x] \in [X]$ *is an S−invariant class if and only if* $[x] = O_a(x)$.

Proof. Suppose $[x] \in [X]$ is an S−invariant and let $y \in O_a(x)$, then $y = s\bar{a}x$ for some $s \in S$. Since $x \in [x]$, then $y \in S([x]) \subset [x]$. Hence $O_a(x) \subset [x]$, and we have $[x] \subset O_a(x)$. Therefore $[x] = O_a(x)$.

Conversely, let $[x] = O_a(x)$ and $y \in S([x])$, then $y = s\bar{a}z$ for some $s \in S$, $z \in [x]$. Hence $z \in O_a(x)$. Take $z = s'\bar{a}x$ for some $s' \in S$. Hence

$y = s\bar{a}z = s\bar{a}(s'\bar{a}x) = ss'\bar{a}x \in O_a(x) = [x]$.

Therefore $[x]$ is S−invariant class.

Theorem 3.3. *Let (S , X , a) be an S−phase flow. Then for all* $x \in X$*, there is a unique S−invariant class* $[y] \subset O_a(x)$.

Proof. For $x \in X$, consider the family of subsets

$Ex = \{z : O_a(z) \subset O_a(x)\}$.

We can define the relation _ on Ex by $x_1 \le x_2$ if $O_a(x_2) \subset O_a(x_1)$ for $x_1, x_2 \in E_x$.

It's clear that the family E_x with $\le$ is a partially order set. Let $\{z_i : i \in \Lambda\}$ be a linearly ordered subset of E_x, where Λ is an index set. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a compact closed subset of X for all $x \in X$. Hence we have a chain $\{O_a(z_i) : i \in \Lambda\}$ of closed subsets of a compact $O_a(x)$. Hence the intersection

$$\bigcap_{i\in\Lambda} O_a(z_i) \neq \emptyset .$$

Take $r \in O_a(z_i)$ for all $i \in \Lambda$. Then $O_a(r) \subset O_a(z_i)$ for all $i \in \Lambda$, imply that $O_a(r)$ is a lower bound of the chain $\{O_a(z_i) : i \in \Lambda\}$ (i.e., r is an upper bound of the linearly order subset $\{z_i : i \in \Lambda\}$ of E_x) . Hence by Zorn's lemma implies that the family E_x has a maximal element, say y. Then $[y] \subset O_a(y) \subset O_a(x)$.

Now, we show that [y] is an S−invariant. Let $z \in O_a(y)$, then $z \in O_a(z) \subset O_a(x)$ and $y \le z$, but by the maximality of y, we get that $z \le y$, this implies $y \in O_a(z)$.Hence $z \in [y]$ (i.e., $O_a(y) \subset [y]$) and we have that $[y] \subset O_a(y)$. Then by Lemma 3.2, [y] is an S−invariant class. Now, let $[\alpha] \neq [y]$ be an S−invariant class such that $[\alpha] \subset O_a(x)$. Then $O_a(\alpha) \subset O_a(x)$. Hence $\alpha \in E_x$. By the maximality of y, we get that $\alpha \le y$, this implies that

$[y] = O_a(y) \subset O_a(\alpha) = [\alpha]$.

Hence $[y] = [\alpha]$, this means that [y] is a unique.

Theorem 3.4. *Let (S , X , a) be an S−phase flow.* $[x] \in [X]$ *has no-return condition for all* $x \in X$.

Proof. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $Oa(x)$ is a compact closed subset of X for all $x \in X$ (i.e., $cl(Oa(x)) = Oa(x)$ for all $x \in X$). Let $z \in Oa(y)$ for some $y \in [x]$ and $Oa(z) \cap [x] \neq \emptyset$. Take $w \in Oa(z)$ and $w \in [x]$. Hence

$x \in Oa(x) \subset Oa(w) \subset Oa(z)$.

On the other hand, $z \in Oa(y)$ for some $y \in [x]$, we have

$z \in Oa(z) \subset Oa(y) \subset Oa(x)$.

Hence $z \in [x]$.

The next theorem clears that if M has the no-return condition , then any a class $[x]$ is entirely contained in M or M^c . Also M is an S−invariant if $[x]$ is an S−invariant class for all $x \in M$.

Theorem 3.5. *Let (S , X , a) be S−phase flow and M be a subset of X has no-return condition.M is an S−invariant set if [x] is an S−invariant class for all* $x \in M$.

Proof. It's clear that $M \subset \bigcup_{x\in M}[x]$ because $x \in [x]$. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a compact closed subset of X for all $x \in X$ (i.e., $cl(O_a(x)) = O_a(x)$ for all $x \in X$). Let $y \in \bigcup_{x\in M}[x]$, then $y \in [x]$ for some $x \in M$. Hence $[x] = [y]$ (i.e., $x \in O_a(y)$ and $y \in O_a(x)$). Since $x \in M$, then $O_a(y) \cap M \neq \emptyset$. By the no-return condition we have t hat $y \in M$. Hence

$M = \bigcup_{x\in M}[x]$

Now, we show that M is an S−invariant set. Let $y \in S(M)$. Then $y = s\bar{a}x$ for some $x \in M$. Hence $y \in O_a(x)$. Since $[x]$ is an S−invariant class then by Lemma 3.2, $[x] = O_a(x)$ and by Equation (1), we get that $y \in [x] \subset M$. Hence M is an S−invariant.

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