Stability Analysis of Discrete-Time Prey-Predator Model with Harvesting Activity and Allee Effect

Harjot Singh, Saminder Talwar, Harkaran Singh

Abstract- In this paper, the stability analysis of discrete-time Prey-Predator model with harvesting activity in presence and absence of Allee effect on prey population has been carried out. Forward Euler method is applied to the continuous model to obtain the discrete-time model. We discussed the stability criterion of the discrete-time model at the fixed points. Numerical simulations have been carried out to show the dynamical behavior of the model.

Keywords: Harvesting activity, Forward Euler method, Critical points, Allee effect. Mathematics Subject Classification 2000: 97M60

I. INTRODUCTION

The Prey-Predator model is a topic of great interest for many mathematicians and ecologists. Many researchers have studied dynamical behavior in ecological modelling particularly in prey-predator system and contributed to growth of continuous models for large size populations [5, 7, 9, 10]. Dhar [3] studied a prey-predator dynamics, where the predator species partially depend upon the prey species in a two patch habitat and obtained the conditions for asymptotic stability therein. Dubey [4] proposed a prey-predator model and observed that the reserve zone has a stabilizing effect on prey-predator interactions. Jeschike et al. [8] presented functional response model that incorporates handling and digesting time of prey and found that predation rate is maximum of either time spent for handling or digesting prey. Wanbia et al. [11] studied the local and global dynamical properties of the positive equilibrium of Lotka-Volterra prey-predator system with distributed delays and shown that if the positive equilibrium does not exist then the equilibrium is globally asymptotic stable and if the positive equilibrium exists then it is locally asymptotically stable. Moghadas et al. [13] extended Gauss-type prey-predator model to include a general monotonic and bounded seasonally varying functional response and investigated the global stability of boundary equilibria and the existence of periodic solutions. Narayan et al. [14] studied a model in which the predator is provided with an alternative feed in addition to the prey, and both the prey and the predator harvested proportional to their population sizes. Many researchers found that the discrete-time models are more appropriate and provide efficient results as compared to the continuous models for small size populations [1, 2, 6, 10].

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In this paper, the stability analysis of discrete-time Prey-Predator model with harvesting activity in presence and absence of Allee effect on prey population has been carried out. In the first stage, Forward Euler method is applied to continuous model with harvesting activity and without Allee effect to obtain the discrete-time model. We discussed the stability criterion of the discrete-time model at fixed points. At the second stage, similar analysis has been carried out by incorporating Allee effect on prey population. Numerical simulations have been carried out to show the dynamical behavior of the model.

II. MATHEMATICAL MODEL

Consider a habitat where prey and predator species are living together. Let $x(t)$ and $y(t)$ are the population of prey and predator respectively at time 't'. Let 'a' is per capita prey growth rate, 'c' is per capita predator mortality rate and 'γ' is the harvesting activity constant. The terms $(-bxy)$ and (dxy) describe the prey-predator encounters which are favorable to predator and fatal to prey respectively. The prey-predator model [12] with harvesting activity is given by

$$\begin{cases} \frac{dx}{dt} = ax - bxy - \gamma x \\ \frac{dy}{dt} = -cy + dxy - \gamma y, \end{cases} \qquad (1)$$

where all the parameters a, b, c, d and γ are positive parameters.

By applying the forward Euler method to system (1), we obtain the discrete-time prey-predator model as follows:

$$\begin{cases} x_{n+1} = x_n + \delta x_n(a - by_n - \gamma) \\ y_{n+1} = y_n + \delta y_n(-c + dx_n - \gamma), \end{cases} (2)$$

where δ is the step size.

The fixed points of the system of equations (2) are $E_1(0,0)$ and $E_2(x_2^*, y_2^*)$ where $x_2^* = \frac{c+\gamma}{d}$, $y_2^* = \frac{a-\gamma}{b}$ and y_2^* exists if $a > \gamma$.

Lemma 2.1: *[see 15] Let $F(\lambda) = \lambda^2 - B\lambda + C$. Suppose that $F(1) > 0$, λ_1 and λ_2 are roots of $F(\lambda) = 0$. Then*

- *(i)* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$;
- *(ii)* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*) if and only if* $F(-1) < 0$;
- *(iii)* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$;
- *(iv)* $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $B \neq 0, 2$;
- *(v)* λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $B^2 - 4C < 0$ *and* $C = 1$.

Let λ_1 and λ_2 are eigen values of jacobian matrix at the critical point $E(x, y)$. Then $E(x, y)$ is called a sink or locally asymptotically stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. $E(x, y)$ is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and

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$|\lambda_2| > 1$). $E(x, y)$ is called a source or locally unstable if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. $E(x, y)$ is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Remark 2.2 (a): *The critical point $E_1(0,0)$ is a sink if $a < \gamma$, saddle if $0 < \gamma < a$ and non-hyperbolic if $\gamma = a$.*

The jacobian matrix of (2) at $E_1(0,0)$ is given by

$$J_1 = \begin{bmatrix} 1+\delta(a-\gamma) & 0 \\ 0 & 1+\delta(-c-\gamma) \end{bmatrix}.$$

The eigen values of jacobian matrix J_1 are $\lambda_1 = 1 + \delta(a-\gamma)$ and $\lambda_2 = 1 + \delta(-c-\gamma)$. Here

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if $a < \gamma$ and $-c < \gamma$ i.e. if $a < \gamma$. Therefore $E_1(0,0)$ is a sink if $a < \gamma$.

(ii) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ if $a > \gamma$ and $-c < \gamma$. Therefore $E_1(0,0)$ is a saddle if $0 < \gamma < a$.

(iii) $|\lambda_1| = 1$ or $|\lambda_2| = 1$ if $a = \gamma$ and $-c = \gamma$. Therefore $E_1(0,0)$ is non-hyperbolic if $\gamma = a$.

Remark 2.2 (b): *The critical point $E_2(x_2^*, y_2^*)$ is a source if $\gamma < a$.*

The jacobian matrix of (2) at $E_2(x_2^*, y_2^*)$ is given by

$$J_2 = \begin{bmatrix} 1 & -b\delta x_2^* \\ \frac{d\delta(a+c-dx_2^*)}{b} & 1 \end{bmatrix}.$$

The corresponding characteristic equation can be written as $\lambda^2 - (trJ_2)\lambda + detJ_2 = 0$,

where

$$trJ_2 = 2 \qquad (3)$$

and

$$detJ_2 = 1 + \delta^2 x_2^* d(a+c-dx_2^*). \quad (4)$$

Let $F(\lambda) = \lambda^2 - (trJ_2)\lambda + detJ_2$. (5)

From (5), we have

$$F(1) = 1 - (trJ_2) + detJ_2. \qquad (6)$$

Using (3) and (4) in (6), we get

$F(1) = \delta^2 x_2^* d(a+c-dx_2^*)$.

As $F(1)$ is positive, we have

$$x_2^* < \frac{a+c}{d}. \qquad (7)$$

From (5), we have

$$F(-1) = 1 + (trJ_2) + detJ_2. \qquad (8)$$

Using (3) and (4) in (8), we get

$F(-1) = 4 + \delta^2 x_2^* d(a+c-dx_2^*)$. Now we have the following cases;

(i) $F(-1) > 0$ if $x_2^* < \frac{a+c+\alpha}{d}$ where $\alpha = \frac{4}{\delta^2 x_2^* d}$ is positive always and $detJ_2 < 1$ if $\frac{a+c}{d} < x_2^*$, which contradicts (7).

(ii) $F(-1) < 0$ if $\frac{a+c+\alpha}{d} < x_2^*$, which contradicts (7).

(iii) $F(-1) > 0$ if $x_2^* < \frac{a+c+\alpha}{d}$ and $detJ_2 > 1$ when $x_2^* < \frac{a+c}{d}$. Therefore $F(-1) > 0$ and $detJ_2 > 1$ if $x_2^* < \frac{a+c}{d}$ i.e. if $\gamma < a$. So $E_2(x_2^*, y^*)$ is a source if $\gamma < a$.

(iv) $F(-1) = 0$ if $x_2^* = \frac{a+c+\alpha}{d}$, which contradicts (7).

(v) $detJ_2 = 1$ when $x_2^* = \frac{a+c}{d}$, which contradicts (7).

Hence the critical point $E_2(x_2^*, y_2^*)$ is a source if $\gamma < a$.

2.3 Numerical simulation

In this section simulation of model (1) has been carried out in the interval [0, 50] taking initial values of x and y in ratio of 5:1.

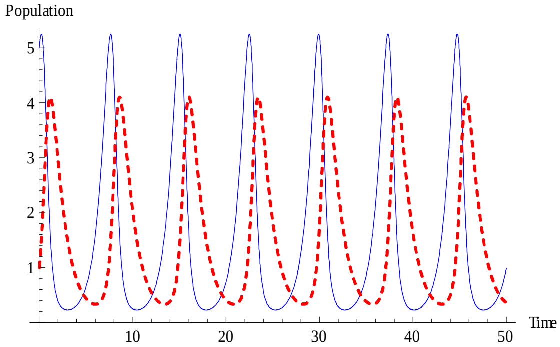

Fig. 2.1a

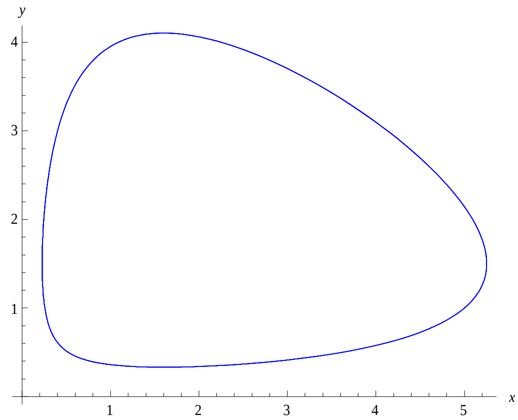

Fig. 2.1b

$a = 1.4, b = 0.8, c = 0.6, d = 0.5, \gamma = 0.2.$

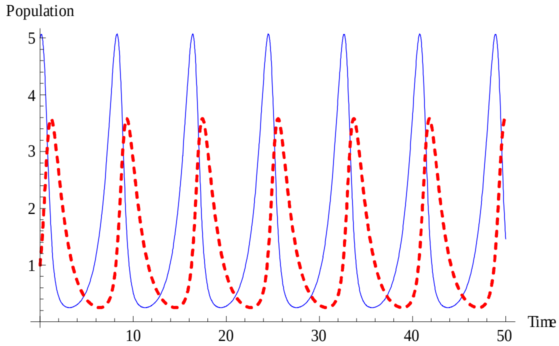

Fig. 2.2a

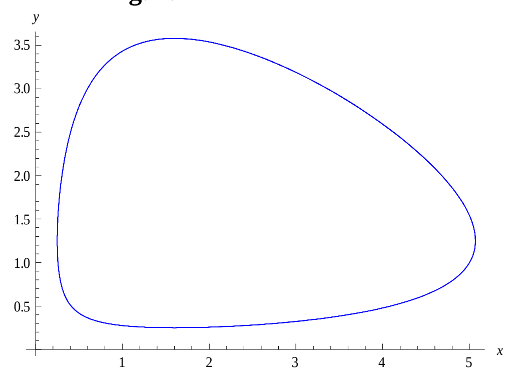

Fig. 2.2b

$a = 1.4, b = 0.8, c = 0.6, d = 0.5, \gamma = 0.4.$

In fig. 2.1a and 2.2a, solid line shows the variation of x with time t and dotted line shows the variation of y with time t. In fig. 2.1b and 2.2b, solid line shows the variation of y with x. It has been observed that the population size of predator decreased when harvesting activity varies from $\gamma = 0.2$ to $\gamma = 0.4$.

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III. ALLEE EFFECT ON PREY POPULATION

The proposed prey-predator model with harvesting activity in the presence of Allee effect on prey population is:

$$\begin{cases} \frac{dx}{dt} = x\left[(a-\gamma)\frac{x}{u+x} - by\right] \\ \frac{dy}{dt} = -cy + dxy - \gamma y. \end{cases} \quad (9)$$

where u is Allee constant with the assumption that $u > 0$.

By applying the forward Euler method to system (9), we obtain the discrete-time model as follows:

$$\begin{cases} x_{n+1} = x_n + \delta x_n\left[(a-\gamma)\frac{x_n}{u+x_n} - by_n\right] \\ y_{n+1} = y_n + \delta y_n(-c + dx_n - \gamma), \end{cases} (10)$$

where $\frac{x_n}{u+x_n}$ is the Allee effect function.

Critical points of the system of equations (9) are $E_3(0,0)$ and $E_4(x_4^*, y_4^*)$ where $x_4^* = \frac{c+\gamma}{d}$ is always positive, and $y_4^* = \frac{(a-\gamma)}{b}\frac{x_4^*}{u+x_4^*}$.

Remark 3.1 (a): *The critical point $E_3(0,0)$ is non-hyperbolic.*

The jacobian matrix of (10) at $E_3(0,0)$ is given by

$J_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The eigen values of jacobian matrix J_3 are $\lambda_1 = 1$ and $\lambda_2 = 1$. Therefore $E_3(0,0)$ is non-hyperbolic.

Remark 3.1 (b): *The critical point $E_4(x_4^*, y_4^*)$ is a source if $\gamma < a$.*

The jacobian matrix of (10) at $E_4(x_4^*, y_4^*)$ is given by

$J_4 = \begin{bmatrix} 1 + (a-\gamma)\delta\beta & -b\delta x_4^* \\ \frac{d\delta(a-\gamma)}{b}\beta(u + x_4^*) & 1 \end{bmatrix}$, where $\beta = \frac{ux_4^*}{(u+x_4^*)^2}$ is always positive.

The corresponding characteristic equation can be written as

$\lambda^2 - (trJ_4)\lambda + detJ_4 = 0,$

where $trJ_4 = 2 + (a-\gamma)\delta\beta$ (11)

and

$detJ_4 = 1 + (a-\gamma)\beta\delta[1 + d\delta x_4^*(u + x_4^*)].$ (12)

Let $F(\lambda) = \lambda^2 - (trJ_4)\lambda + detJ_4$. (13)

From (13), we have

$F(1) = 1 - (trJ_4) + detJ_4.$ (14)

Using (11) and (12) in (14), we get

$F(1) = (a-\gamma)\beta\delta^2 x_4^*(u + x_4^*).$

As $F(1)$ is positive, we have $\gamma < a$. (15)

From (13), we have

$F(-1) = 1 + (trJ_4) + detJ_4.$ (16)

Using (11) and (12) in (16), we get

$F(-1) = 4 + (a-\gamma)\beta\delta[2 + \delta x_4^*(u + x_4^*)].$

Now we have the following cases;

(i) $F(-1) > 0$ if $a > \gamma - l$, where $l = \frac{4}{\beta\delta[2+\delta x_4^*(u+x_4^*)]}$ and $detJ_4 < 1$ when $a < \gamma$ which contradicts (15).

(ii) $F(-1) < 0$ if $a < \gamma - l$ which contradicts (15).

(iii) $F(-1) > 0$ if $a > \gamma - l$ and $detJ_4 > 1$ if $a > \gamma$. Therefore $F(-1) > 0$ and $detJ_4 > 1$ if $\gamma < a$. Therefore $E_4(x_4^*, y_4^*)$ *is a source if* $\gamma < a$.

(iv) $F(-1) = 0$ if $a = \gamma - l$ which contradicts (15).

(v) $detJ_4 = 1$ if $a = \gamma$ which contradicts (15).

Hence the critical point $E_4(x_4^*, y_4^*)$ is a source if $\gamma < a$.

3.2 Numerical simulations

In this section simulation of model (9) has been carried out in the interval [0, 50] taking initial values of x and y in ratio of 5:1.

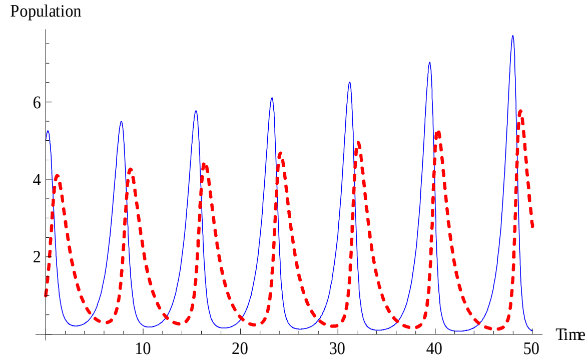

Fig. 3.1a

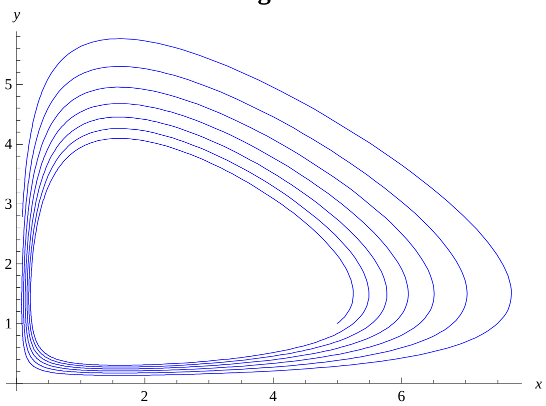

Fig. 3.1b

$a = 1.4, b = 0.8, c = 0.6, d = 0.5, \gamma = 0.2$ and $u = 0.010$

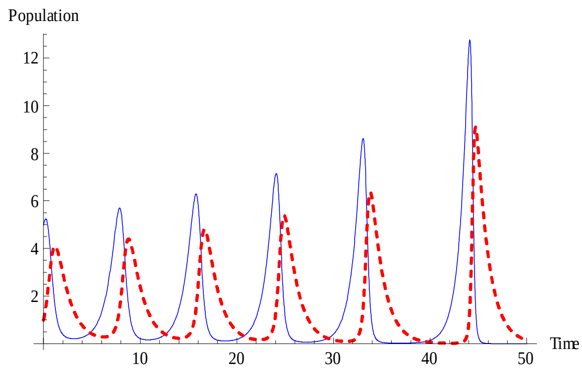

Fig. 3.2a

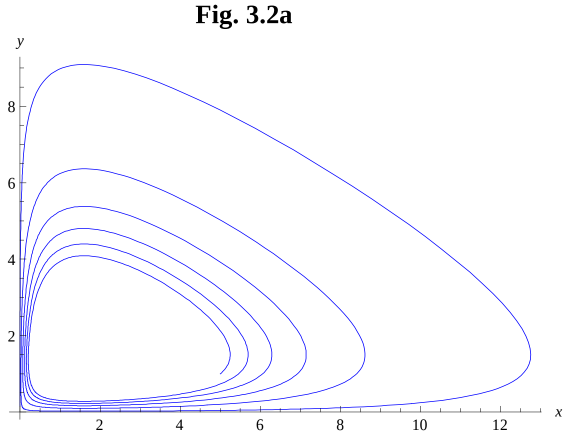

Fig. 3.2b

$a = 1.4, b = 0.8, c = 0.6, d = 0.5, \gamma = 0.2$ and $u = 0.018$

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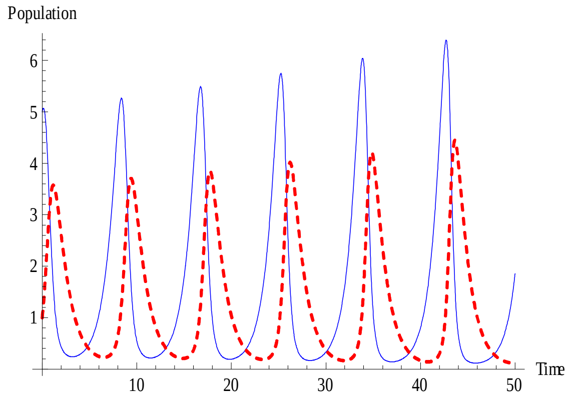

Fig. 3.3a

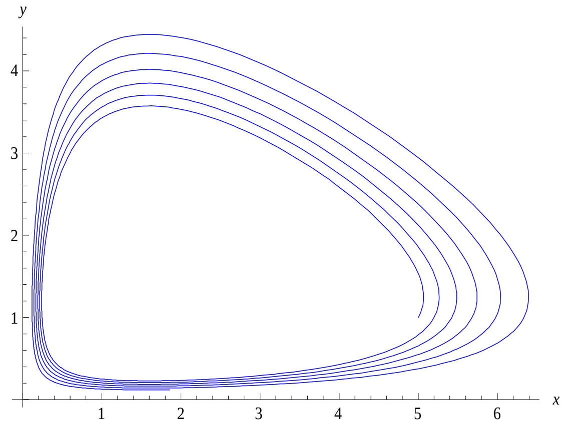

Fig. 3.3b

$a = 1.4, b = 0.8, c = 0.6, d = 0.5, \gamma = 0.4$ and $u = 0.010$

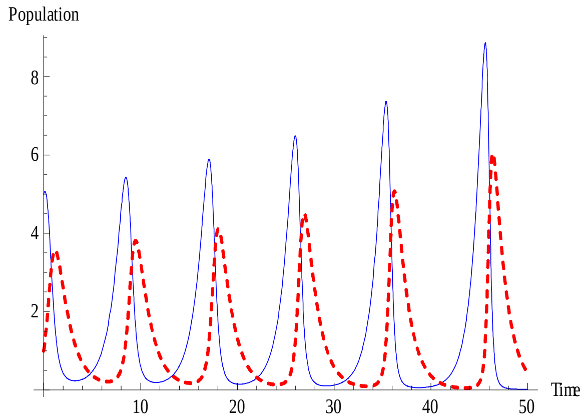

Fig. 3.4a

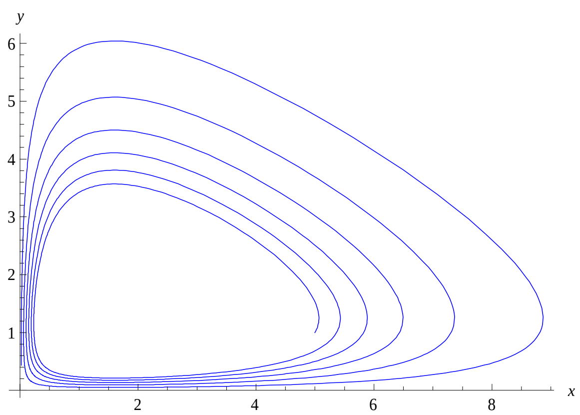

Fig. 3.4b

$a = 1.4, b = 0.8, c = 0.6, d = 0.5, \gamma = 0.4$ and $u = 0.018$

In fig. 3.1a, 3.2a, 3.3a and 3.4a solid line shows the variation of x with time t and dotted line shows the variation of y with time t. In fig. 3.1b, 3.2b, 3.3b and 3.4b solid line shows the variation of y with x. It has been observed that the populations of predator as well as prey increases progressively with the passage of time. By critical analysis of all figures, it has been observed that the time interval between two consecutive maxima increased for both prey and predator populations and less number of maxima observed in same time intervals as we increase Allee constant 'u' from 0.010 to 0.018.

Further, it has been observed that there is continuous increase in population size of prey and predator by increasing the Allee constant 'u'.

IV. CONCLUSIONS

In the prey-predator model with harvesting activity and without Allee effect, $E_2(x_2^*, y_2^*)$ is a source if $\gamma < a$. In the prey-predator model with harvesting activity in the presence of Allee effect on prey population, $E_4(x_4^*, y_4^*)$ is a source if $\gamma < a$.

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