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The Extension of the Riemann's Zeta Function

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***Abstract:** In mathematics, the search for exact formulas giving all the prime numbers, certain families of prime numbers or the n-th prime number has generally proved to be vain, which has led to contenting oneself with approximate formulas [8]. The purpose of this article is to give a new proof of the Riemann hypothesis [4]-which is closely related to the distribution of prime numbers- by y introducing* $\hat{S}$ *a new extension of the of the Riemann zeta function*

***Keywords:** Prime Number, number theory, distribution of prime numbers, the law of prime numbers, the Gamma function, the Mertens function, quantum mechanics, black Holes, holomorphic function, Hilbert-Polya's conjecture, the Riemann hypothesis*

In memory of the great professor, the physicist and mathematician, Moshé Flato.

I. INTRODUCTION, RECALL, NOTATIONS AND DEFINITIONS

Prime numbers [See 3, 4, 5, 6, 7, 8] are used especially in information technology, such as public-key cryptography which relies on factoring large numbers into their prime factors. And in abstract algebra, prime elements and prime ideals give a generalization of prime numbers. In mathematics, the search for exact formulas giving all the prime numbers, certain families of prime numbers or the n-th prime number has generally proved to be vain, which has led to contenting oneself with approximate formulas [8]. Recall that Mills' Theorem [8]: "There exists a real number A, Mills' constant, such that, for any integer n > 0, the integer part of A^{3^n} is a prime number" was demonstrated in 1947 by mathematician William H. Mills , assuming the Riemann hypothesis [4, 5, 6,7] is true. Mills' Theorem [8] is also of little use for generating prime numbers. Recall that a link has been established between the prime numbers, the zeros of the Riemann zeta function and the energy level of various **quantum systems** [see 1 and 2] The purpose of this article is to to give a new proof of the Riemann hypothesis [4]. by y introducing $\hat{S}$ a new extension of the of the Riemann zeta function.

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II. THE PROOF OF THE RIEMANN HYPOTHESIS

Theorem: The real part of every nontrivial zero of the Riemann zeta function is 1/2.

The link between the function ζ and the prime numbers had already been established by Leonhard Euler with the formula [5], valid for $\Re(s)>1$:

$$\zeta(s)=\prod_{p\in P}\frac{1}{1-p^{-s}}=\frac{1}{(1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s})\ldots}$$

where the infinite product is extended to the set P of prime numbers. This formula is sometimes called the Eulerian product.

And since the Dirichlet eta function can be defined by $\eta(s)=(1-2^{1-s})\zeta(s)$ where: $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$

We have in particular: $\zeta(z)=\frac{1}{1-2^{1-z}}\frac{\sum_{n=1}^{\infty}(-1)^{n-1}}{n^z}$

for $0<\Re(z)<1$

Let $s=x+iy$, with $0<\Re(s)<1$

$$\zeta(s)\zeta(s)=\prod_{p\in P}\frac{1}{1-p^{-s}}\frac{1}{1-p^{-s}}=\prod_{p\in P}\frac{1}{(1-e^{-x\ln(p)}\cos(y\ln(p)))^2+(e^{-x\ln(p)}\sin(y\ln(p)))^2}$$

But : $\prod_{p\in P}\frac{1}{(1-e^{-x\ln(p)}\cos(y\ln(p)))^2+(e^{-x\ln(p)}\sin(y\ln(p)))^2}\geq\prod_{p\in P}\frac{1}{(1+e^{-x\ln(p)})^2+(e^{-x\ln(p)})^2}$

If $\zeta(s)=0$, then $\prod_{p\in P}\frac{1}{(1+e^{-x\ln(p)})^2+(e^{-x\ln(p)})^2}=0$ and since the non-trivial zeros of $\zeta(s)=0$ are symmetric with respect to the line $X=\frac{1}{2}$ because the zeta function satisfies the functional equation [4, 6] :

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

then $x=\frac{1}{2}+\alpha$, and if $s'=\frac{1}{2}-\alpha+iy$, then $\zeta(s')=0$

But the function $\frac{1}{(1+e^{-t\ln(p)})^2+(e^{-t\ln(p)})^2}$ is increasing in $[0,1]$, so $\prod_{p\in P}\frac{1}{(1+e^{-t\ln(p)})^2+(e^{-t\ln(p)})^2}=0$ $\forall\ t\in[\frac{1}{2}-\alpha,\frac{1}{2}+\alpha]$

As $\prod_{p\in P}\frac{1}{(1+e^{-z\ln(p)})^2+(e^{-z\ln(p)})^2}$ is holomorphic: because:

$\prod_{p\in P}\frac{1}{(1+e^{-z\ln(p)})^2+(e^{-z\ln(p)})^2}=\prod_{p\in P}\frac{1}{1-A/p^z}\frac{1}{1-B/p^z}$ with $A=i-1$ and $B=-i-1$, and both $\prod_{p\in P}\frac{1}{1-A/p^z}$ and $\prod_{p\in P}\frac{1}{1-B/p^z}$ are holomorphic in $\{z\in\mathbb{C}\setminus\{1\},\Re(z)\geq\frac{1}{2}\}$

as we have:

$\prod_{p\in P}\frac{1}{1-A/p^z}=\prod_{p\in P}1+f_p(z)$ with $f_p(z)=\frac{1}{(p^z/A)-1}$.

$|f_p(z)|\leq\frac{1}{|p^z/A|-1}=\frac{1}{(p^{\Re(z)}/\sqrt{2})-1}\leq\frac{k}{p^{\frac{1}{2}}}$, where k is a positive real constant.

So $\left|\sum_{p\in P,p=N}^{\infty}f_p(z)\right|\leq k\left|\sum_{p=N}^{\infty}\frac{1}{n^{\frac{1}{2}}}\right|=k\left|\zeta_N(\frac{1}{2})\right|$

But (see Lemma 1 [6]): $\zeta_N(\frac{1}{2})=o_N(1)$

We deduce that the series $\sum_p|f_p|$ converges normally on any compact of $\{z\in\mathbb{C}\setminus\{1\},\Re(z)\geq\frac{1}{2}\}$ and consequently $\prod_{p\in P}\frac{1}{1-A/p^z}$ is holomorphic in

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$\{z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}\}$

In the same way $\prod_{p \in P} \frac{1}{1 - B/p^z}$ is holomorphic in $\{z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}\}$

If $\alpha \neq 0$, then the holomorphic function $\prod_{p \in P} \frac{1}{(1+e^{-z\ln(p)})^2 + (e^{-z\ln(p)})^2}$ will be null (because null on $]\frac{1}{2}, \frac{1}{2} + \alpha]$), and it follows that $\prod_{p \in P} \frac{1}{1 - A/p^z}$ or $\prod_{p \in P} \frac{1}{1 - B/p^z}$ is null in $\{z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}\}$. Let's show that this is impossible:

If $\prod_{p \in P} \frac{1}{1 - A/p^z} = \prod_{p \in P} 1 + f_p(z) = 0$ with $f_p(z) = \frac{1}{(p^z/A) - 1}$. $\forall\ z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}$. So for the same reason as above, the application:

$\hat{S}$: $X \rightarrow \prod_{p \in P} \frac{1}{1 - X/p^z}$ is holomorphic in the open quasi-disc $D = \{X \in \mathbb{C}, 0 < |X| < \sqrt{2}\}$ with $z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}$

Let's extend the function $\hat{S}$ by setting:

For $z \in \mathbb{C} \setminus \{1\}, \Re(z) > \frac{1}{2}$ and $\forall\ s \in \mathbb{R}$ with $s \leq 0$ such as $\Re(s+z) \geq 0$

$\hat{S}(C/q^s) = \prod_{p \in P} \frac{1}{1 - C/(q^s p^z)}$ (where q is a prime number, and C is such that $|C| = \sqrt{2}$)

In particular we have:

$\hat{S}(A/q^s) = \prod_{p \in P} \frac{1}{1 - A/(q^s p^z)}$ (where q is a prime number)

But for $z \in \{z \in \mathbb{R} \setminus \{1\}, z > \frac{1}{2}\}$ we have:

$$\prod_{p \in P} \left| \frac{1}{1 - A/(q^s p^z)} \right| \leq \prod_{p \in P} \left| \frac{1}{1 - A/(p^z)} \right|$$

It follows that:

$$\hat{S}(A/q^s) = 0$$

So :

$$\hat{S}(X) = 0, \forall\ X \in D$$

And consequently:

$$\hat{S}(1)(z) = \zeta(z) = 0 \quad \forall\ z \in \{z \in \mathbb{C} \setminus \{1\}, \Re(z) > \frac{1}{2}\}$$

which is absurd, so $\alpha = 0$, hence the Riemann hypothesis.

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III. CONCLUSION

By considering the Riamann zeta function as an action on the **particles** of the complex plane, I showed that on the positive half plane, the zeta action can only be canceled on the line $x=\frac{1}{2}$.

The idea was to study the square modulus of the zeta function. And assuming that the function vanishes at a point in the band $0<\Re(z)<1$ and outside the line $x=\frac{1}{2}$, by symmetry of the roots the square function of zeta module - which is a real function- will cancel on two reals a and b, and by the growth of the square function of the zeta module on [a, b], the square function of the module will cancel on [a, b]. But by extending the square function of the zeta module by a holomorphic function, the latter must be zero because the roots of a non-zero holomorphic function are separated. And by decomposing this last function into two holomorphic functions, one of them will be zero and will imply that the zeta function will be zero, which is absurd.

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Authors Contributions	I am only the sole author of the article.

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AUTHOR PROFILE

Mohamed Sghiar holds a DEA in mathematics, is an independent researcher, and a contractual professor at the rectorate of the Dijon academy. Passionate about both physics and mathematics, he has taken a particular interest in the problem of the Riemann hypothesis, which is closely related to quantum mechanics. This explains why the proofs in most of his articles have a physical aspect.

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