Continuous Monotonic Decomposition of Some Standard Graphs by using an Algorithm

C. Sujatha, A. Manickam

Abstract: In this paper we elaborate an algorithm to compute the necessary and sufficient conditions for the continuous monotonic star decomposition of the bipartite graph $K_{n,r}$ and the number of vertices and the number of disjoint sets. Also an algorithm to find the tensor product of $P_n \times P_1$ has continuous monotonic path decomposition. Finally we conclude that in this paper the results described above are complete bipartite graphs that accept Continuous monotonic star decomposition. There are many other classes of complete tripartite graphs that accept Continuous monotonic star decomposition. In this research article Extended to complete m-partite graphs for greater values of m. Also the algorithm can be developed for the tensor product of different classes such as $C_n, W_n, K_{1,n}$ with $P_n$.

Keywords: Complete bipartite graph, Continuous monotonic star decomposition, Tensor product.

Mathematical subject classification: 03B52, 03E72, 08A72.

1. INTRODUCTION

A simple graph with the property that there is a path between every pair of vertices is known as a connected graph. The degree of a vertex $v$ of any graph is the number of edges incident with $v$ and is denoted by $d(v)$ and the distance between the two vertices $u$ and $v$ of $G$ is the length of the shortest $v-u$ path in $G$ and is denoted by $d(u,v)$. $\{G_i / i = 1,2,...,n\}$ be a collection of edge-disjoint sub graphs of $G$ such that $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_n)$, then the collection $\{G_i\}$ is called a decomposition of $G$. If each $G_i$ is connected and $|E(G_i)| = i$ for each $i = 1,2,...,n$, then it is called a continuous monotonic decomposition of $G$. A complete graph with vertices $n \in \mathbb{N}$, denoted by $K_n$ is a connected simple graph with every vertex is connected with every other vertex by an edge [1]. A graph with $n$ vertices $v_1,v_2,...,v_n$, where $n \geq 3$, and edges $\{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\},...,\{v_{n-1},v_n\},\{v_n,v_1\}$ is known as a cycle, $C_n$.

II. STAR DECOMPOSITION OF A GRAPH

Let $G = (V, E)$ be a connected simple graph of order $m$ and size $n$. If a decomposition $G_1,G_2,...,G_n$ of $G$ is said to be a continuous monotonic decomposition (CMD) if each $G_i$ is connected and $|E(G_i)| = i \forall i \in \mathbb{N}$. [2] Introduced Ascending Sub graph Decomposition (ASD) as a decomposition of $G$ into subgraphs $G_i$ (not necessarily connected) and is isomorphic to a proper sub graph $G_{i+1}$. [3] Introduced a new concept known as continuous monotonic decomposition of Graphs [3]. If $G$ admits a CMD, $\{G_1,G_2,...,G_n\} \forall n \in \mathbb{N}$, where each $G_i$ is a star, then we say that $G$ admits Continuous Monotonic Cycle Decomposition (CMCD)[4].

III. CONTINUOUS MONOTONIC DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS $K_{m,n}$

A graph $G$ is a bipartite graph if $V(G)$ can be partitioned into two subsets $U$ and $V$, called bipartite sets such that every edge of $G$ joins a vertex of $u$ and a vertex of $v$. In a bipartite graph, if every vertex of $u$ is adjacent to every vertex of $v$, then such graph is called complete bipartite graph. A complete bipartite graph with $|U| = s$ and $|V| = t$ is denoted by $K_{s,t}$. If either $s = 1$ or $t = 1$ then $K_{s,t}$ is a star. Continuous Monotonic Decomposition of a wide variety of graphs had been studied by [4,5].

IV. TENSOR PRODUCT

For two graphs $G$ and $H$, the tensor product $G \times H$ has vertex set $V(G) \times V(H)$.
Continuous Monotonic Decomposition of Some Standard Graphs by using an Algorithm

4.1. Theorem
Let \( G \) be a connected simple graph of order \( r \) and size \( s \). Then \( G \) admits a CMD \( H_{1}, H_{2}, \ldots, H_{n} \) if and only if \( q = (n+1)C_{2} \).

4.2. Theorem
\( K_{n}^{+} \) admits a Continuous monotonic star decomposition for all \( n \geq 1 \).

4.3. Theorem
(i) \( K_{n,2n+1} \) admits Continuous monotonic star decomposition for \( n \geq 1 \)
(ii) \( K_{n+1,2n+1} \) admits Continuous monotonic star decomposition for \( n \geq 1 \)

4.4. Theorem
There is an edge decomposition of \( G \) such that each partition class is a star in \( G \) if and only if \( G \) is bipartite.

4.5. Lemma
Let \( G \) be a edge disjoint union of stars \( S_{i+1}, S_{i+2}, \ldots, S_{i+k} \) for some \( k > 0 \) such that \( q = i(i+1)/2 \). Then can be decomposed into stars \( S_{1}, S_{2}, \ldots, S_{j} \).

4.6. Theorem
The complete bipartite graph \( K_{m,r}(m \leq r) \) can be decomposed into stars \( S_{1}, S_{2}, \ldots, S_{2n} \) (CMSD) if and only if \( m = n - 1 \) and \( r = 2n + 1 + j \) where \( i, j > 0 \) such that \( n = i(j+1)/(j-2i) \).

4.6.1. Algorithm
1. \((K_{m,r})\)
Step 1: Initially enter the values \( m, r, x \)
Step 2: Read the value of \( x \)
Step 3: for \( i = 1 \) to \( n \) do
Step 4: for \( j = 1 \) to \( n \) do
(i) if \( (j > 2 * i) \) then
(ii) compute \( y = i(j+1)/(j-2i) \)
(iii) compute \( m = y - i \)
(iv) compute \( r = 2 * y + 1 + j \)
Step 5: if \( (m \leq r \& \& m > 0) \)
Step 6: Print \( i, j, y, m \) and \( r \)
Step 7: Go to step -3 until \( i > n \)
Step 8: Stop;
Step 9: End
Output
Enter the value of \((n=15)\)

4.7. Continuous monotonic decomposition of Tensor Product of \( P_{n} \times P_{s} \)

4.7.1. Theorem
For any integer \( n \), \( P_{n} \times K_{2} \) has a CMD \( \{H_{1}, H_{2}, \ldots, H_{m}\} \) if and only if there exist an integer \( m \) satisfying the following properties.
(i) \( m = 4k \) or \( m = 4k - 1 \) \((k \geq 1, k \in \mathbb{Z})\)
(ii) \( m(m + 1)/2 = 4n - 4 \)

4.7.2. Lemma
(i) Let \( m \equiv 0 \) \((\text{mod } 4)\). Two copies of the set \( \{1,2,\ldots,m\} \) can be partitioned into four sets \( S_{1}, S_{2}, S_{3} \) and \( S_{4} \) such that \( \Sigma_{a \in S_{1}} a = \Sigma_{b \in S_{2}} b = \Sigma_{c \in S_{3}} c = \Sigma_{d \in S_{4}} d = n - 1 \). Here \( m(m + 1) = 4n - 4 \).
(ii) Let \( m + 1 \equiv 0 \) \((\text{mod } 4)\). Two copies of the set \( \{1,2,\ldots,m\} \) can be partitioned into four sets \( S_{1}, S_{2}, S_{3} \) and \( S_{4} \) such that \( \Sigma_{a \in S_{1}} a = \Sigma_{b \in S_{2}} b = \Sigma_{c \in S_{3}} c = \Sigma_{d \in S_{4}} d = n - 1 \). Here \( m(m + 1) = 4n - 4 \).

4.7.3. Theorem
For any integer \( n \), \( P_{n} \times K_{2} \) has two copies of continuous monotonic decomposition \( \{H_{1}, H_{2}, \ldots, H_{m}\} \) if and only if there exist an integer \( m \) satisfying the following properties.
(i) \( m = 4k \) or \( m = 4k - 1 \) \((k \geq 1, k \in \mathbb{Z})\)
(ii) \( m(m + 1) = 4n - 4 \)

Proof
Let \( G = P_{n} \times P_{3} \). By definition

Table 1

<table>
<thead>
<tr>
<th>( I )</th>
<th>( J )</th>
<th>( 2n )</th>
<th>( M )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>8</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>4</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>24</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>14</td>
<td>5</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>8</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>6</td>
<td>1</td>
<td>21</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>48</td>
<td>21</td>
<td>56</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>20</td>
<td>7</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>12</td>
<td>3</td>
<td>26</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>80</td>
<td>36</td>
<td>90</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>44</td>
<td>18</td>
<td>55</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>32</td>
<td>12</td>
<td>44</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>26</td>
<td>9</td>
<td>39</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>20</td>
<td>6</td>
<td>35</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>120</td>
<td>55</td>
<td>132</td>
</tr>
<tr>
<td>6</td>
<td>13</td>
<td>168</td>
<td>78</td>
<td>182</td>
</tr>
<tr>
<td>6</td>
<td>14</td>
<td>90</td>
<td>39</td>
<td>105</td>
</tr>
</tbody>
</table>
\( |E(G)| = 4n = 4\). Assume \( P_n \times P_3 \) has two copies of CMD \( \{H_1, H_2, \ldots, H_m\} \). We have
\[
|E(G)| = 2x^{(n+1)}C_2. \quad \text{Here } 4n - 4 = 2x^{(n+1)}C_2.
\]
Since \( P_n \times P_3 \) has two copies of CMD, \( 4n - 4 = 2 \times (1 + 2 + \cdots + m) = 2 \times m(m + 1)/2 = m(m + 1) \).

Hence \( m(m + 1) = 4k(K \geq 1, kcZ) \). Now either \( m = 4k \) or \( m = 4k - 1 \).

Conversely assume \( m(m + 1) = 0 \) (mod 4). Let \( G = P_n \times P_3 \). Let \( P_n = (u_1, u_2, \ldots, u_n) \),
\( P_3 = (v_1, v_2, v_3) \).

Define \( w_{ij} = (u_i, v_j) \), where \( 1 \leq i \leq n, 1 \leq j \leq 3 \).

Case 1: Suppose \( n \) is odd.

Define
\[
T_1 = \{w_{i1}, w_{i+1-j} : 1 \leq i \leq n - 1, i - odd\}
\]
\( \cup \{w_{i2}, w_{i+1-j} : 1 \leq i \leq n - 1, i - even\} \).
\[
T_2 = \{w_{i2}, w_{i+j} : 1 \leq i \leq n - 1, i - odd\}
\]
\( \cup \{w_{i3}, w_{i+1-j} : 1 \leq i \leq n - 1, i - even\} \).
\[
T_3 = \{w_{i3}, w_{i+j} : 1 \leq i \leq n - 1, i - odd\}
\]
\( \cup \{w_{i4}, w_{i+j} : 1 \leq i \leq n - 1, i - even\} \).

Here \( |T_1| = |T_2| = |T_3| = |T_4| = n - 1 \). Also \( |T_1| + |T_2| = (m-1)C_2 \) and \( |T_3| + |T_4| = (m+1)C_2 \).

By Lemma 10, \( \{1, 2, \ldots, m\} = S_1 \cup S_2 \) and \( \{1, 2, \ldots, m\} = S_3 \cup S_4 \).

Decompose
\[
T_1, T_2 \text{ into trees } \{H_i\} \text{ as follows: } T_1 = Y_{icS_1} H_i \text{ and } T_2 = Y_{icS_2} H_i. \quad \text{Clearly two copies of } \{H_1, H_2, \ldots, H_m\} \text{ forms the continuous monotonic decompositions of } P_n \times P_3.
\]

Case 2: Suppose \( n \) is even.

Define \( T_1, T_2, T_3 \) and \( T_4 \). As the case 1, two copies of \( \{H_1, H_2, \ldots, H_m\} \) forms the continuous monotonic decompositions of \( P_n \times P_3 \).

### 4.7.3.1. Algorithm

**Step 1**: Initially enter the values \( n, m, a, b \)

**Step 2**: Read the values of \( m, n \)

**Step 3**: While \( (a \leq m - 1) \)

**Step 4**: Store \( a = 1; b = 2 \); (i) For \( i = 1 \) to \( n \) do

(ii) Line \((i, a) - (i + 1, b)\)

(iii) Swap \((a, b)\)

**Step 5**: Store \( a = 2; b = 1 \)

(i) For \( i = 1 \) to \( n \) do

(ii) Line \((i, a) - (i + 1, b)\)

(iii) Swap \((a, b)\)

**Step 6**: Goto Step 3 Until \( (a \leq m - 1) \)

**Step 7**: Stop

**Step 8**: End

**Output.**

<table>
<thead>
<tr>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>( T_3 )</th>
<th>( T_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>From</td>
<td>To</td>
<td>From</td>
<td>To</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>(2, 2)</td>
<td>(1, 2)</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>(3, 1)</td>
<td>(2, 1)</td>
<td>(3, 2)</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>(4, 2)</td>
<td>(3, 2)</td>
<td>(4, 1)</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>(5, 1)</td>
<td>(5, 1)</td>
<td>(6, 1)</td>
</tr>
<tr>
<td>(5, 1)</td>
<td>(6, 2)</td>
<td>(5, 2)</td>
<td>(6, 1)</td>
</tr>
<tr>
<td>(6, 2)</td>
<td>(7, 1)</td>
<td>(6, 1)</td>
<td>(7, 2)</td>
</tr>
<tr>
<td>(7, 1)</td>
<td>(8, 2)</td>
<td>(7, 2)</td>
<td>(8, 1)</td>
</tr>
<tr>
<td>(8, 2)</td>
<td>(9, 1)</td>
<td>(9, 1)</td>
<td>(10, 1)</td>
</tr>
<tr>
<td>(9, 1)</td>
<td>(10, 2)</td>
<td>(10, 2)</td>
<td>(11, 1)</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>(11, 1)</td>
<td>(11, 1)</td>
<td>(12, 1)</td>
</tr>
<tr>
<td>(11, 1)</td>
<td>(12, 2)</td>
<td>(12, 2)</td>
<td>(13, 1)</td>
</tr>
<tr>
<td>(12, 2)</td>
<td>(13, 1)</td>
<td>(13, 1)</td>
<td>(14, 1)</td>
</tr>
<tr>
<td>(13, 1)</td>
<td>(14, 2)</td>
<td>(14, 2)</td>
<td>(15, 1)</td>
</tr>
<tr>
<td>(14, 2)</td>
<td>(15, 1)</td>
<td>(15, 1)</td>
<td>(16, 1)</td>
</tr>
</tbody>
</table>
Continuous Monotonic Decomposition of Some Standard Graphs by using an Algorithm

V. CONCLUSION

In this paper we described above complete bipartite graphs that accept Continuous monotonic star decomposition. There are many other classes of complete tripartite graphs that accept Continuous monotonic star decomposition. Finally we conclude that in this paper can be extended to complete m-partite graphs for greater values of m. Also the algorithm can be developed for the tensor product of different classes such as $C_n, W_n, K_{1,n}$ with $P_n$.

CONFLICT OF INTEREST

The author confirms that there no conflict of interest to declare for this publications.

ACKNOWLEDGEMENT

The author would like to express his sincere thanks to the Editor-in-Chief for this valuable suggestions towards improvisation of the article.

REFERENCES