International Journal of Basic Sciences and Applied Computing (IJBSAC)
ISSN: 2394-367X, Volume-2 Issue-7, June 2019

Spline Solution of Linear Seventh Order Boundary Value Problems using Tenth Degree Functions and Comparison with Different Degrees of Spline Solutions

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***Abstract:** In this communication numerical solutions of general linear boundary value problems of order seven are considered. Tenth degree spline approximations are developed following Cubic Spline Bickley's procedure and applied. Approximate numerical solutions are computed at different step lengths, and also absolute errors are calculated. The results are tabulated and pictorially illustrated. Further, the results of the tenth degree spline function solutions are compared with eighth and ninth degree spline solutions.*

***Keywords:** Spline approximations; seventh order boundary value problems; tenth degree spline; numerical results.*

I. INTRODUCTION

Boundary value problems (BVPs) are used as mathematical models in a wide variety of disciplines including biology, physics, and engineering [3]. The theory of seventh order boundary value problems is not much available in the numerical analysis literature. The seventh order boundary value problems generally arise in modeling induction motors with two rotor circuits, the behavior of such models show up in the seventh order differential equation model by Richards and Sarma, [12]. Akram and Siddiqi [2], presented the solution of seventh order boundary value problem using octic spline. Siddiqi and Akram [13] presented the solutions of fifth and sixth order boundary value problems using non-polynomial spline. [11], used the variation of parameters method for solving the fifth order boundary value problems. Akram and Rehman [1], solved the fifth order boundary value problems using the Reproducing Kernel space method. In [4], Numerical solution of boundary-value and initial Boundary-value problems is found by using spline functions. Schoenberg [5, 6] given detailed analysis of splines. Convergence properties of the spline fit has been discussed in [7] by Ahlberg, E. N. Nilson. Variation of parameters method is used for solving sixth-order boundary value problems in [9].

Generally, boundary value problems arise in the Mathematics, Physics and Engineering Sciences. Over the years, there are several authors who studied these boundary value problems. There are several methods, such as Finite differences, Orthogonal splines, Galarkin, Collocation, Finite element method etc to discrete the boundary value problems. The reference [8] contain theorems which detail the conditions for existence and uniqueness of solutions of such BVPs.

In this work we construct tenth degree splines by using Bickley's method and apply it to the linear seventh boundary value problems. The work has been illustrated through examples with different step lengths.

1.1. Construction of Tenth Degree Spline Function

Assume that the interval $[x_{0,}x_n]$ is divided in to n sub intervals with grid points $x_0, x_1, x_2, x_3, \dots x_n$ and the function $u(x)$ in the interval $[x_{0,}x_1]$, is represented by tenth degree spline in the form

$$s(x) = a + b(x-x_0) + c(x-x_0)^2 + d(x-x_0)^3 + g(x-x_0)^4 + j(x-x_0)^5 + k(x-x_0)^6 + l(x-x_0)^7 + t(x-x_0)^8 + v(x-x_0)^9 + w_0(x-x_0)^{10}. \quad (1)$$

For the next interval $[x_{1,}x_2]$ we add a term $w_1(x-x_1)^{10}$ to the spline function, for the interval $[x_{2,}x_3]$ we add another term $w_2(x-x_2)^{10}$and so on until we reach x_n hence the function $s(x)$ represents in the form

$$s(x) = a + b(x-x_0) + c(x-x_0)^2 + d(x-x_0)^3 + g(x-x_0)^4 + j(x-x_0)^5 + k(x-x_0)^6 + l(x-x_0)^7 + t(x-x_0)^8 + v(x-x_0)^9 + \sum_{i=0}^{n-1} w_i\ (x - xi10. \quad (2)$$

and its seventh derivative is

$$s^{(7)}(x) = 5040l + 40320t(x-x_0) + 181440v(x - x02+604800i=0n-1wi\,x-xi3 \quad (3)$$

1.2. Method of Obtaining the Solution of Seventh Order Boundary Value Problem Using Tenth Degree Spline Function

Consider the linear seventh order differential equation

$$y^{(7)}(x) + f(x)y(x) = r(x), \quad (4)$$

with boundary conditions

$$y(x_0) = \beta,\quad y(x_n) = \gamma,\quad (x_0) = \beta',\quad y'(x_n) = \gamma',\quad y''(x_0) = \beta'',\quad y''(x_n) = \gamma'',\quad y'''(x_0) = \beta''', \quad (5)$$

From (5), and taking spline approximation in (4) at $x = x_i\ for\ i = 0,1,2,3,4,5,6,\dots,n$, we get $(n+8)$ equations in $(n+10)$ unknowns $a, b, c, d, g, j, k, l, t, v, w_0, w_1, w_2, w_3, w_4, \dots, w_{n-1}$.

Revised Manuscript Received on May 22, 2019.

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After determining these unknowns we substitute them in (2) and thus get the tenth degree spline approximation of $y(x)$. Putting $x = x_1, x_2, x_3, \dots x_n$ in the spline function thus determined we get the solution at the grid points; substituting (2) and (3) in the equation (4) at $x = x_m$ where $m = 0,1,2,3,\dots n$ and $s(x) = r(x)$, we get the following

$$af(x_m) + bf(x_m)(x_m - x_0) + cf(x_m)(x_m - x_0)^2 + df(x_m)(x_m - x_0)^3 + gf(x_m)(x_m - x_0)^4 + jf(x_m)(x_m - x05+kfxmxm-x06$$
$$+l[f(x_m)(x_m - x_0)^7 + 5040] + t[f(x_m)(x_m - x_0)^8 + 40320xm-x0+vfxmxm-x09+181440xm-x02+i=0n-1wi\,fxmxm-xi10+ 604800xm-xi3=sxm. \quad (6)$$

From the boundary conditions (5) we obtain

$$y(x_0) = \beta,$$
$$\beta = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 + l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + \sum_{i=0}^{n-1} w_i\ (x - xi10,$$
$$a = \beta. \quad (7)$$
$$y(x_n) = \gamma,$$
$$a + b(x_n - x_0) + c(x_n - x_0)^2 + d(x_n - x_0)^3 + g(x_n - x_0)^4 + j(x_n - x_0)^5 + k(x_n - x_0)^6 + l(x_n - x07+txn -x08+vxn -x09+i=0n-1wi\,xn -xi\,10 =\gamma. \quad (8)$$

$$y'(x_0) = \beta',$$
$$\beta' = b + 2c(x - x_0) + 3d(x - x_0)^2 + 4g(x - x_0)^3 + 5j(x - x_0)^4 + 6k(x - x_0)^5 + 7l(x - x_0)^6 + 8t(x - x_0)^7 + 9v(x - x_0)^8 + 10\sum_{i=0}^{n-1} w_i\ (x - x_i)^9,$$
$$b = \beta'. \quad (9)$$
$$y'(x_n) = \gamma',$$
$$b + 2c(x_n - x_0) + 3d(x_n - x_0)^2 + 4g(x_n - x_0)^3 + 5j(x_n - x_0)^4 + 6k(x_n - x_0)^5 + 7l(x_n - x_0)^6 + 8t(x_n - x_0)^7 + 9v(x_n - x_0)^8 + 10\sum_{i=0}^{n-1} w_i\ (x_n - x_i)^9 = \gamma'. \quad (10)$$
$$y''(x_0) = \beta'',$$
$$\beta'' = 2c + 6d(x - x_0) + 12g(x - x_0)^2 + 20j(x - x_0)^3 + 30k(x - x_0)^4 + 42l(x - x_0)^5 + 56t(x - x_0)^6 + 72v(x - x_0)^7 + 90\sum_{i=0}^{n-1} w_i\ (x - x_i)^8,$$
$$2c = \beta''. \quad (11)$$
$$y''(x_n) = \gamma'',$$
$$\gamma'' = 2c + 6d(x - x_0) + 12g(x - x_0)^2 + 20j(x - x_0)^3 + 30k(x - x_0)^4 + 42l(x - x_0)^5 + 56t(x - x_0)^6 + 72v(x - x_0)^7 + 90\sum_{i=0}^{n-1} w_i\ (x - x_i)^8,$$
$$2c + 6d(x_n - x_0) + 12g(x_n - x_0)^2 + 20j(x_n - x_0)^3 + 30k(x_n - x_0)^4 + 42l(x_n - x_0)^5 + 56t(x_n - x_0)^6 + 72v(x_n - x_0)^7 + 90\sum_{i=0}^{n-1} w_i\ (x - xi\,8=\gamma''. \quad (12)$$
$$y'''(x_0) = \beta''',$$
$$\beta''' = 6d + 24g(x - x_0) + 60j(x - x_0)^2 + 120k(x - x_0)^3 + 210l(x - x_0)^4 + 336t(x - x_0)^5 + 504v(x - x_0)^6 + 720\sum_{i=0}^{n-1} w_i\ (x - xi7,$$
$$6d = \beta'''. \quad (13)$$

The coefficient matrix of the unknowns, $w_{n-1}, w_{n-2}, w_{n-3}, w_{n-4}, \dots, w_2, w_1, w_0, v, t, l, k, j, g, d, c, b, a,$ in these equations are in the Hessenberg form. From the boundary conditions and taking spline approximation in the linear seventh order boundary value problem at $x = x_i$ for $i = 0,1,2,3,4,5,\dots,n$ in $(n+8)$ equations we find the unknowns by assuming that $w_{n-1} = w_{n-2} = w_{n-3}$. After determining these unknowns, are substituted in (2) and putting $x = x_1, x_2, x_2, x_4, x_5, \dots, x_n$ in the spline function thus determined, we get the solution at the grid points.

II. NUMERICAL RESULTS

We consider two linear boundary value problems, their numerical solution and absolute errors are found at different step lengths. The approximate solution, exact solution and absolute errors at the grid points are summarized in tabular form and shown graphically. The relationship of highest absolute errors at different step lengths has been existing in tabular form.

Problem 1. Consider the linear seventh order boundary value problem with constant coefficients

$$u^{(7)}(x) = -u(x) - e^x(35 + 12x + 2x^2), \qquad 0 \le x \le 1, \quad (14)$$

with boundary conditions

$$u(0) = 0, \quad u'(0) = 1,\ u^{(2)}(0) = 0, u^{(3)}(0) = -3$$
$$u(1) = 0, \quad u'(1) = -e, \quad u^{(2)}(1) = -4e. \quad (15)$$

The exact solution of the problem *is* $u(x) = x(1-x)e^x$

Approximating $u(x)$ with

$$s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 + l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + \sum_{i=0}^{n-1} w_i\ (x - x_i)^{10}$$

The boundary conditions becomes

$$s(0) = 0,\ \ s'(0) = 1,\ \ s''(0) = 0 \text{ and } s'''(0) = -3$$

With the Seventh derivative of $s(x)$, (14) is given by

$$s^{(7)}(x) = -s(x) - e^x(35 + 12x + 2x^2). \quad (16)$$

from the boundary conditions, we get $a = 0$, $b = 1$, $c = 0$, and $d = -0.5$,

and the spline fuction $s(x)$ reduces to the form

$$s(x) = (x - x_0) - 0.5(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 + l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + \sum_{i=0}^{n-1} w_i\ (x - xi10. \quad (17)$$

Solution with $h = 0.2$

Since $h = 0.2$ the grid points are $x_0, x_1, x_2, x_3, x_4, x_5$ where, $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$ and

$$s(x) = (x - x_0) - 0.5(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 - 0.006944(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + \sum_{i=0}^{4} w_i\ (x - x_i)^{10}. \quad (18)$$

Solving equation (18) we get the following values for unknown coefficients

$g = -0.33333$ $w_0 = -0.0000331811$ $w_1 = 0.00000347$ $w_2 = -0.00001136$ $k = -0.03333214$ $w_3 = -0.00001136$ $w_4 = -0.00001136$ $t = -0.00119734$ $l = -0.0069444$ $j = -0.1250002$ $v = -0.000159548$

Substituting these values in equation (17) we get the spline approximation $f(x)$ of $u(x)$.

The values of $s(x)$, $u(x)$ and the corresponding absolute errors at x_1, x_2, x_3, x_4 have been given in the Table 1 and the comparison has been shown in Figure 1.

Table 1: Numerical solution $s(x)$, exact solution $u(x)$ and absolute error of the problem1 with $h = 0.2$

x	s(x)	u(x)	Absolute error
0.2	0.195424441238452	0.1954244413056	6.717462697203E − 011
0.4	0.358037926880896	0.3580379274339	5.530090274596E − 010
0.6	0.437308511249002	0.4373085120937	8.447200161576E − 010
0.8	0.356086548221534	0.3560865485587	3.3726099690767E − 010
1.0	−1.13104899345e − 014	0.00000000000	1.1310489934555E − 014

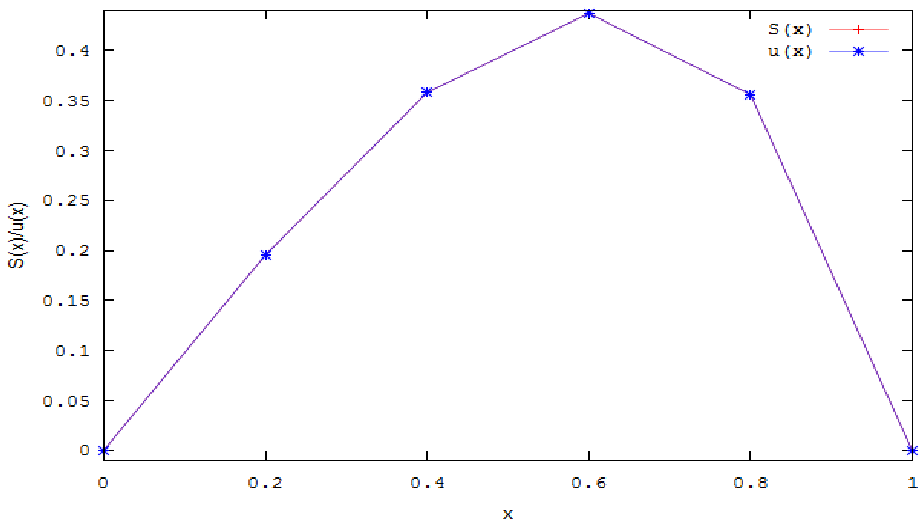

Figure 1. Comparison of approximate solution and exact solution for problem 1 with $h = 0.2$

Solution with $h = 0.1$

Given that $h = 0.1$ so the grid points are x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} where, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$, $x_5 = 0.5$, $x_6 = 0.6$, $x_7 = 0.7$, $x_8 = 0.8$, $x_9 = 0.9$, $x_{10} = 1$ and

$$s(x) = (x - x_0) - 0.5(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 -0.006944(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + \sum_{i=0}^{9} w_i\ (x - x_i)^{10}. \qquad (19)$$

Solving equation (19) we get the values for unknown coefficients

Substituting these values in equation (19) we get the spline approximation $s(x)$ of $u(x)$. The values of $s(x)$, $u(x)$ and the corresponding absolute errors at the grid points have been given in the Table 2 and the comparison has been shown in Fig 2.

Table 2: Numerical solution $s(x)$, exact solution $u(x)$ and absolute error of the problem1 with $h = 0.1$

x	s(x)	u(x)	Absolute error
0.1	0.0994653826190540	0.0994653826268083	9.02899977006655E-012
0.2	0.195424441218261	0.195424441305627	8.73660033207102E-011
0.3	0.283470349338641	0.283470349590961	2.52319998228501E-010

0.4	0.358037926984394	0.358037927433905	4.49510983990820E-010
0.5	0.412180317056123	0.412180317675032	6.18908979621580E-010
0.6	0.437308511367867	0.437308512093722	7.25854987226882E-010
0.7	0.422888067802474	0.422888068568800	7.66326002654694E-010
0.8	0.356086547786610	0.356086548558795	7.72185038133699E-010
0.9	0.221364279221554	0.221364280004125	7.82571007995614E-010

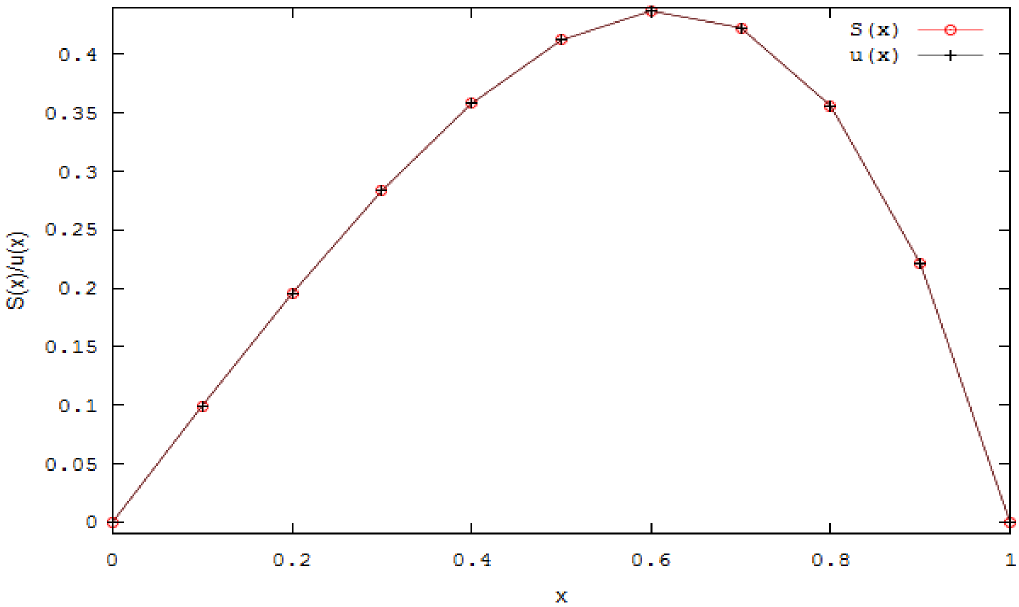

Figure 2. Comparison of approximate solution and exact solution for problem 1 with $h = 0.1$

The maximum absolute errors at these step length are $8.447200E-010$ and $7.8257100E-010$ respectively. From this we understand that there is good agreement with the exact solution

Problem 2 The following seventh order linear boundary value problem with variable coefficients is considered.

$$u^7(x) = xu(x) + e^x(x^2 - 2x - 6), \quad 0 \le x \le 1 \tag{20}$$

with the boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1, \quad u^3(0) = -2,$$
$$u(1) = 0, \quad u'(1) = -e, \quad u''(1) = -2e, \tag{21}$$

The exact solution of (20) is $u(x) = (1-x)e^x$

Table 3: Numerical solution using tenth degree spline approximation *s(x)*, exact solution *u(x)* and absolute errors of problem 2 with $h = 0.2$

x	$S(x)$	$u(x)$	$Absolute\ error$
0. 2	0.977122206521071	0.977122206528136	7.06534830641203E − 012
0.4	0.895094818530000	0.895094818584762	5.47620837565432E − 011
0.6	0.728847520074112	0.728847520156204	8.20919998645309E − 011
0.8	0.445108185666043	0.445108185698494	3.2450986342524E − 011

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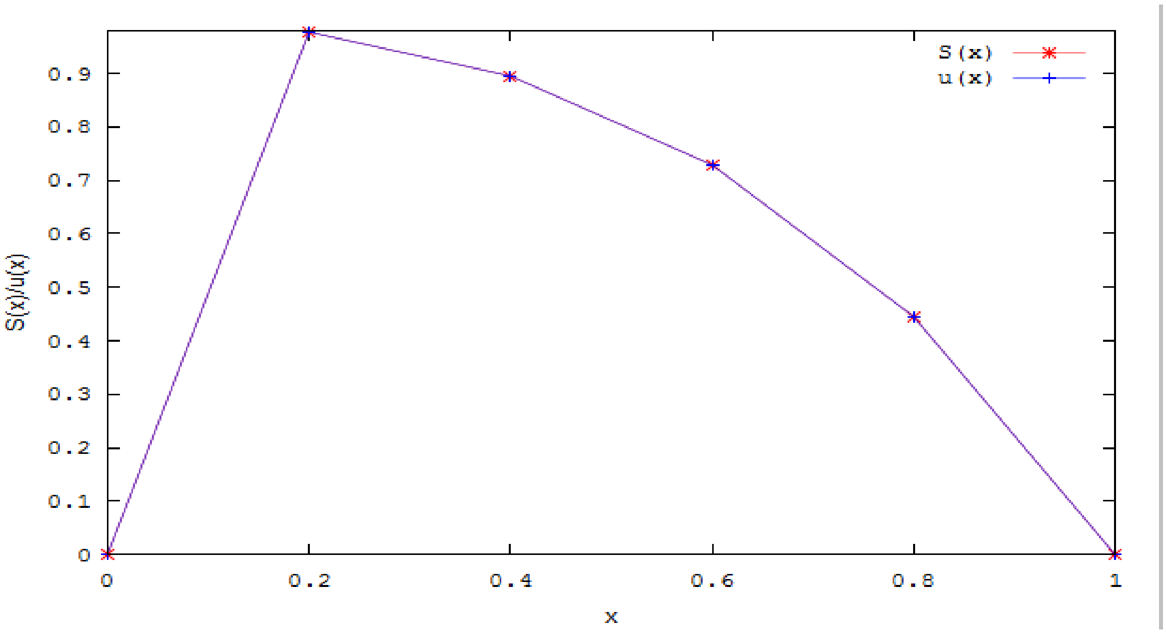

Figure 3. Comparison of approximate solution and exact solution for problem 2 with $h = 0.2$

Table 4: Numerical solution using tenth degree spline approximation *s(x)*, exact solution *u(x)* and absolute errors of problem 2 with $h = 0.1$

x	S(x)	u(x)	Absolute error
0.1	0.994653826269063	0.994653826268083	$9.7999386383662E-013$
0.2	0.977122206538923	0.977122206528136	$1.0787037929560E-011$
0.3	0.944901165332005	0.944901165303202	$2.8802960017060E-011$
0.4	0.895094818630416	0.895094818584762	$4.5654258151728E-011$
0.5	0.824360635403098	0.824360635350064	$5.3034354685621E-011$
0.6	0.728847520203677	0.728847520156204	$4.7473025510669E-011$
0.7	0.604125812272944	0.604125812241143	$3.1800895250455E-011$
0.8	0.445108185712051	0.445108185698494	$1.3557044376000E-011$
0.9	0.245960311116585	0.245960311115695	$8.8973273193460E-013$

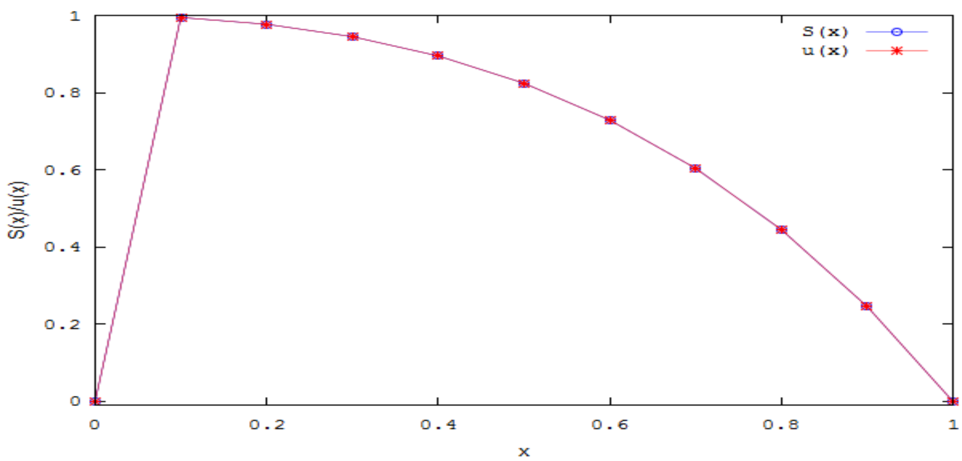

Figure 4. Comparison of approximate solution and exact solution for problem 2 with $h = 0.1$

Table 4 represnts the approximate solution,exact solution and absolute errors of problem 2 at $h = 0.1$ and comparison has given in figure 4 .the maximum absolute error is $5.303435e - 011$

2.1. Comparative study of tenth degree with eighth and ninth degree spline functions

The numerical results obtained by tenth degree spline approximation are compared with the numerical results obtained by eighth and ninth degree spline approximation [10, 11] at different step lengths. Comparison is given in tabular form and shown graphically.

Table 5: Comparison for absolute errors of eighth, ninth, tenth and eleventh degree spline approximation for problem 1 at h = 0.2

x	*Exact solution*	*Absolute error for* 8^{th} *degree*[10]	*Absolute error for* 9^{th} *degree* [11]	*Absolute error for* 10^{th} *degree*
0.2	0.1954244441305627	$5.000000e - 08$	$3.74400e - 09$	$6.7174626972e - 011$
0.4	0.358037927433905	$4.016999e - 07$	$3.36110e - 08$	$5.5300902746e - 010$
0.6	0.437308512093722	$6.220000e - 07$	$7.56990e - 08$	$8.44720016158e - 010$
0.8	0.356086548558795	$2.538000e - 07$	$6.03450e - 08$	$3.37260996908e - 010$

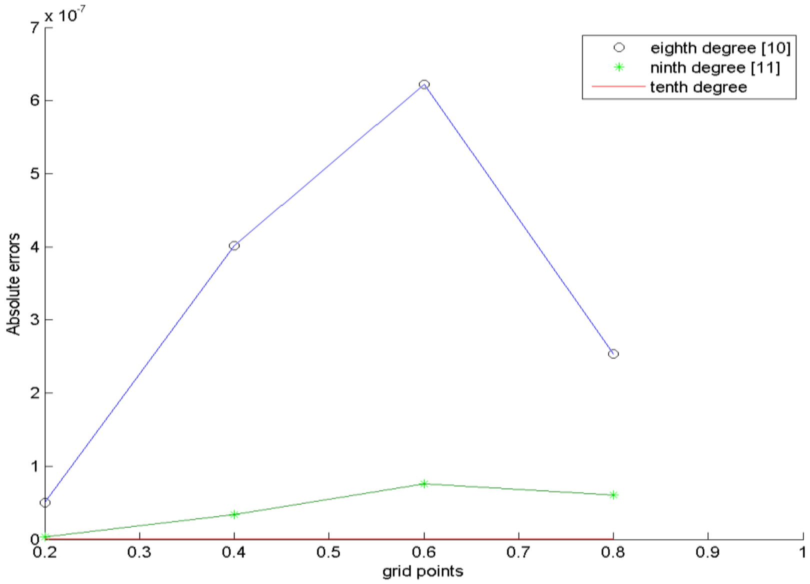

Figure 5. Comparison for absolute errors of eighth, ninth and tenth degree spline approximation for problem 1 at h = 0.2

Table 6: Comparison for absolute errors of eighth, ninth and tenth degree spline approximation for problem 1 at $h = 0.1$

x	Exact solution	Absolute error for 8^{th} degree[10]	Absolute error for 9^{th} degree[11]	Absolute error for 10^{th} degree

0.1	0.09946538262	$7.9999E-08$	$3.3200E-10$	$9.02899977E-012$
0.2	0.19542444130	$9.8000E-07$	$3.5540E-09$	$8.73660033E-011$
0.3	0.283470349	$2.3657E-05$	$1.2366E-08$	$2.523199982E-010$
0.4	0.358037927	$7.4400E-06$	$2.7224E-08$	$4.495109839E-010$
0.5	0.412180317	$1.1289E-05$	$4.3750E-08$	$6.189089796E-010$
0.6	0.437308512	$1.3459E-05$	$5.4648E-08$	$7.258549872E-010$
0.7	0.422888068	$1.4430E-05$	$5.1837E-08$	$7.663260026E-010$
0.8	0.356086547	$2.0369E-05$	$3.3095E-08$	$7.721850381E-010$
0.9	0.221364280	$4.7770E-05$	$8.5130E-09$	$7.825710079E-010$

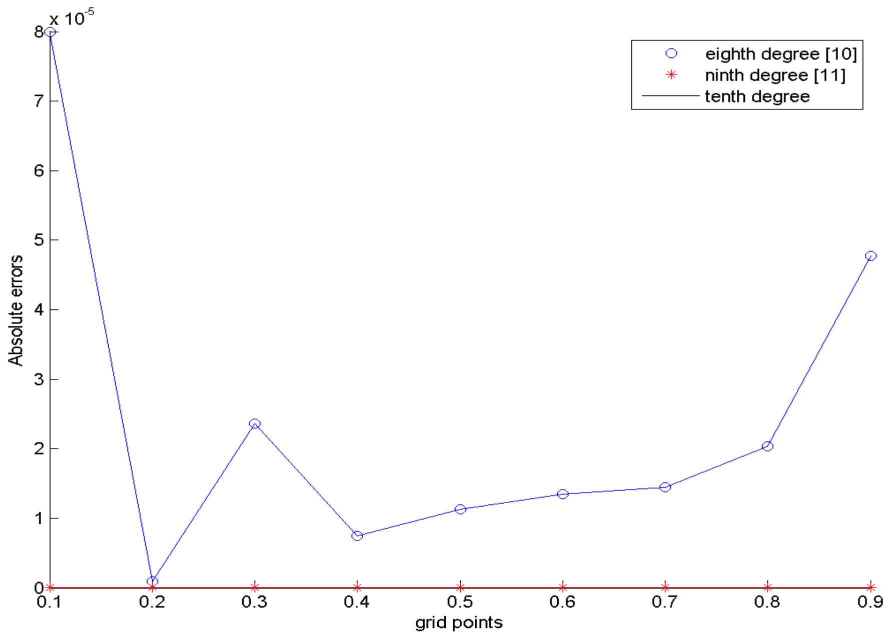

Figure 6. Comparison for absolute errors of eighth, ninth and tenth degree spline approximation for problem 1 m at $h = 0.1$

Table 7: Comparison for absolute errors of eighth, ninth and tenth degree spline approximation for problem 2 at $h = 0.2$

x	*Exact solution*	*Absolute error for* 8^{th}*degree*[10]	*Absolute error for* 9^{th}*degree*[11]	*Absolute error for* 10^{th}*degree*
0.2	0.9771222	$1.2000E-08$	$8.6590E-09$	$7.0653483E-012$
0.4	0.89509481	$8.6000E-08$	$8.3135E-08$	$5.47620837E-011$
0.6	0.72884752	$1.8500E-06$	$1.8903E-07$	$8.20919998E-011$
0.8	0.44510818	$2.0500E-06$	$1.6752E-07$	$3.24509863E-011$

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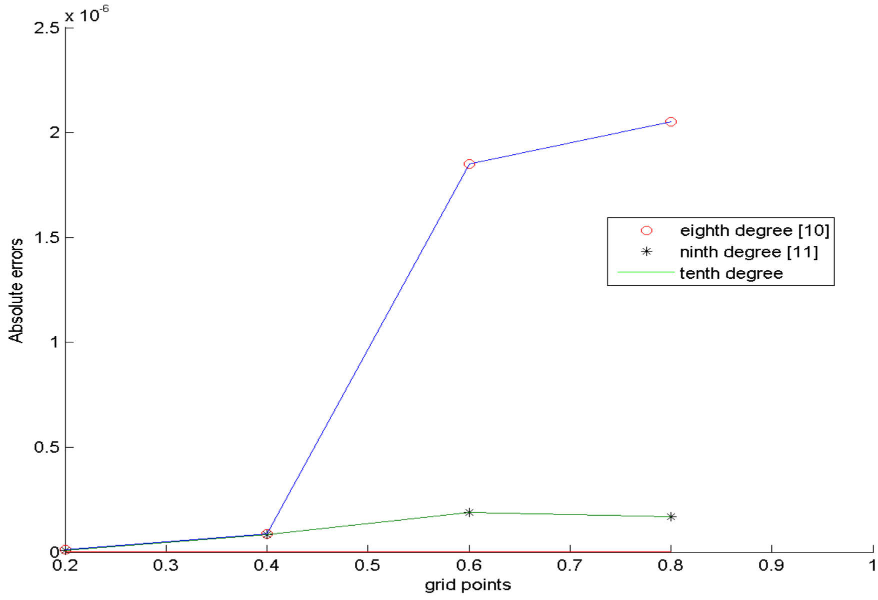

Figure 7. Comparison for absolute errors of eighth, ninth and tenth degree spline approximation for problem 2 at $h = 0.2$

Table 8: Comparison for absolute errors of eighth, ninth, tenth and eleventh degree spline approximation for problem 2 at $h = 0.1$

x	Exact solution	Absolute error for 8^{th}degree [10]	Absolute error for 9^{th}degree[11]	Absolute error for 10^{th}degree
0.1	0.99465382	$7.99990E-09$	$9.000E-10$	$9.79993863E-013$
0.2	0.97712220	$7.87990E-08$	$1.410E-10$	$1.07870379E-011$
0.3	0.944901165	$2.32699E-07$	$4.871E-10$	$2.88029600E-011$
0.4	0.89509481	$3.85500E-07$	$3.937E-09$	$4.56542581E-011$
0.5	0.82436063	$4.10000E-07$	$9.280E-10$	$5.30343546E-011$
0.6	0.72884752	$1.03900E-07$	$5.709E-09$	$4.74730255E-011$
0.7	0.60412581	$3.87499E-0.7$	$6.239E-08$	$3.1800895E-011$
0.8	0.44510818	$8.12699E-07$	$1.298E-08$	$1.35570443E-011$
0.9	0.24596031	$7.56099E-07$	$1.048E-08$	$8.89732731E-013$

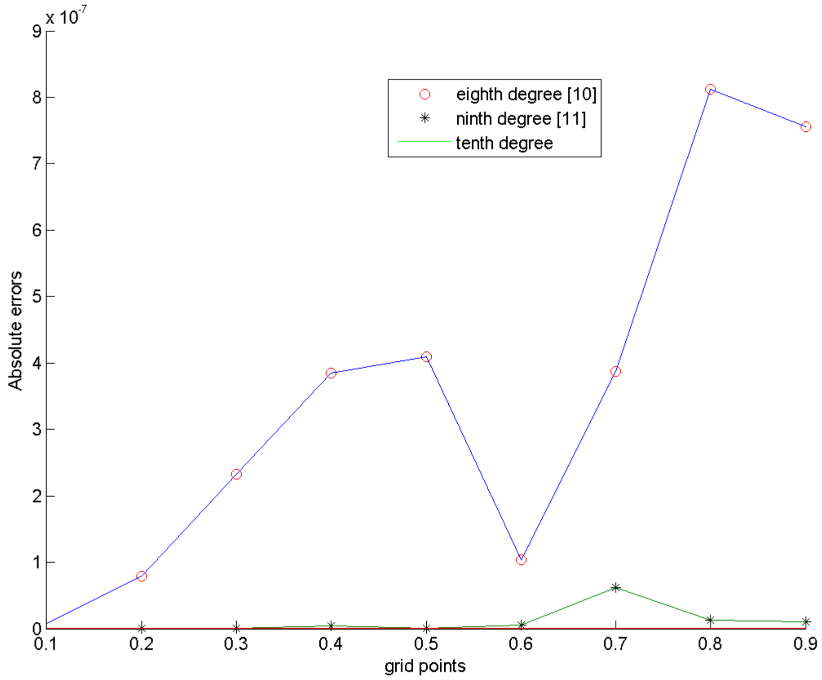

Figure 8. Comparison for absolute errors of eighth, ninth and tenth degree spline approximations for problem 2at $\mathbf{h = 0.1}$

III. CONCLUSION

We developed the numerical method to obtain the solution of seventh order linear boundary value problems using tenth degree spline. Tenth degree spline approximation has been employed on two problems at different step lengths. The computational work has been carried out using MATLAB software. Numerical solution of the problem has been found with $h = 0.2$ and $h = 0.1$. Approximate solution, exact solution and absolute errors with $h = 0.2$ and $h = 0.1$ of the problem1 and 2 are summarized in the Tables 1-4 respectively. The comparison has been shown in Figures 1-4 respectively.

The minimum absolute errors or the maximum accuracy at these step length are 6.7174×10^{-11}, and 9.0289×10^{-12} for problem1 at $h = 0.2$ and $h = 0.1$ and 7.06534×10^{-12}, and 9.79993×10^{-13} for problem2 at $h = 0.2$ and $h = 0.1$ respectively. From this we understand that there is good agreement with the exact solution. It is observed that the absolute errors are less than the errors occur in the methods [10, 11]. It is also observed that our proposed method is well suited for the solution of higher order boundary value problems and reduces the computational work. Spline approximation method converges to exact solutions more rapidly as the degree increases and if the step length decreases the numerical solution is more accurate.

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