Monoid Algebras and Application

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Abstract*: *Algebras as, free algebras over posets, upper-lattice Boolean algebras and semigroup algebras can be viewed as F2-algebras over a monoid or a quasi-monoid. This paper deals with this notion.

Index Terms:* *Monoid algebras, quasi-monoid algebras, Boolean algebras, Tail algebras, upper-lattice algebras, free algebras over posets, interval algebras.

I. Introduction

In the theory of Boolean algebras, see for instance **[K]**, generally we consider a Boolean algebra $\mathbf{B}$ as a Boolean lattice $\langle \mathbf{B}, \mathbf{0}, \mathbf{1}, \wedge, \vee, - \rangle$ where $\langle \mathbf{B}, \wedge, \vee \rangle$ is a distributive lattice with a first element $\mathbf{0}$ and a last element 1, and each member $\mathbf{b} \in \mathbf{B}$ has a unique complement $-\mathbf{x}$, that is $\mathbf{x} \vee (-\mathbf{x}) = \mathbf{1}$ and $\mathbf{x} \wedge (-\mathbf{x}) = \mathbf{0}$. But we can see a Boolean algebra as a structure of the form $\langle \mathbf{B}, \mathbf{0}, \mathbf{1}, +, . \rangle$, where $\mathbf{x} + \mathbf{y} = \mathbf{x} \triangle \mathbf{y} := (\mathbf{x} \vee \mathbf{y}) - (\mathbf{x} \wedge \mathbf{y})$ is the symmetric difference and x· y = x∧y is an algebra over the field $\mathbf{k} = \mathbb{F}_2 = \mathbb{Z} / (2)$, see L. Heindorf [H1] and E. Evans [E].

The scope of this work is: To extend all their results in the context of k-algebras, see for example [H1], [E] and to establish some properties of algebras over monoid (or semi group algebras) as algebras then derive some concrete representations (free poset algebras R. Bonnet and M. Rubin [BR], R. Bonnet, L. Faouzi and W. Kubis [BLK], R. Bonnet, U. Abraham, W. Kubi´s and M. Rubin [BAKR], upper semi lattice algebras M. Bekkali and D. Zhani [BZ1,BZ2], I. Chakir and M. Pouzet [CP]), see J. D. Monk [Monk]. This is possible since in these algebras over monoid there is a normal form of non zero elements which makes it easy to look at them as algebra over the field $\mathbf{k} = \mathbb{F}_2 = \mathbb{Z} / (2)$ see M. Bekkali and D. Zhani [BZ1,BZ2]. It has to be noticed that the Boolean algebraic character remains hidden since the use of complement in these structures stands out for itself. Precise statements of relevant theorems needed for this paper is at the end of the article.

On the other hand, the class of quasi-monoid k-algebras is studied; one classical example of monoid k-algebra is the ring of polynomials k[X], which is the k-algebra over the monoid $\langle \mathbb{N}, + \rangle$.

At the end, . we study the special examples developed by Heindorf [H1] and Evans [E], which are called semi-ring algebras or quasi-upper-lattice algebras, that are Boolean algebras of the form $\mathbb{F}_2[\widehat{M}]$ where $\widehat{M}$ is a quasi-monoid. These examples were initiated by Koppelberg and Monk [KM], developed by Heindorf [H1], Evans in [E] and by Bekkali and Zhani in [BZ1,BZ2].

Recall the definition of a disjunctive set.

Definition 1.1. *A subset H of a Boolean algebra A is disjunctive whenever:*

(1) *0 is not in H;*

(2) *If $h, h_1, \dots, h_n$ are in H with n > 0 and $h \leq h_1 + \cdots + h_n$, then $h \leq h_i$ for some i.*

Definition 1.2. *A Boolean algebra A is a semi-group algebra whenever it is generated by a set H which has the following properties:*

(1) $0,1 \in H$

(2) *H is closed under multiplication*

(3) $H \setminus \{0\}$ *is disjunctive.*

Notice the reminiscent similarity in this definition and the definition of a basis, below, in (†). Moreover, elementary properties of semigroup algebras are found in Chapter 2 of cardinal functions book [Monk]. The notion was introduced by Heindorf in Fundamenta vol. 135, pp. 37 − 47.

Now, in what follows, k denotes a commutative field, M , N , L, etc. denote monoids. Recall that a monoid (also called semigroup) is a set (M ; ·) with " · " is commutative, associative and has a unit denoted by 1^{M}. We shall denote $\mathrm{x} \cdot^{\mathrm{M}} \mathrm{y}$ by $\mathrm{x} . \mathrm{y}$ or more simply by xy, and x(yz) = (xy)z by xyz. Now, recall that (S; ·) is called a semi-lattice whenever " · " is commutative, associative, and $\mathbf{x}^2 = \mathbf{x}$ for all x ∈ S; the binary operation · denoted by ∧ satisfies $x \leq y$ whenever $\mathrm{x} \wedge \mathrm{y} = \mathrm{x}$. Hence, a semi-lattice (S, ·), with a greatest element, is a monoid.

Next, let *k*, *M* be a field and a monoid respectively. The *k*-algebra over *M* , denoted by $k[M]$, may be defined as follows: Consider the k-vector space $k^{(M)}$ having M as k-basis. Now the multiplication · on *M*, extended by biliearity, induces a structure on $k^{(M)}$ of k-algebra. This algebra, denoted by $k[M]$ is called the k-algebra over the monoid *M* . For instance, a construction may be achieved as follows: Denote by $k^{(M)}$ the subspace of k^M consisting of all $x \in k^M$ such that $\sigma(x) := \{s \in M : x_s \neq 0\}$ is finite. Hence the canonical basis of $k^{(M)}$ is $\{e_s : s \in M\}$ where $e_s = \langle \delta_{s,t} : t \in S \rangle$ where $\delta_{s,s} = 1$ and $\delta_{s,t} = 0$ for $t \neq s$. The multiplication of $k^{(M)}$ is defined by $e_s . e_t := e_{st}$ extended by bilinearity: see [Bo, Ch III, §2.6]), [L, Ch 5, §1] and [R, §1.2].

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The algebra $k^{(M)}$ is denoted by $k[M]$, and for simplicity, we identify s and e_s. Notice that,

(†) M is basis of the algebra $k[M]$ and M is closed under product.

In particular $st \neq 0$ for $s, t \in M$.

Note that $k[M]$ is a unitary commutative algebra (the unit$1^{k[M]}$ of $k[M]$ is 1^M). A monoid algebra is a k-algebra over some monoid. A well-known example of monoid k-algebra is the algebra of polynomials. Indeed, for a given set I, $k[(X_i)_{i\in I}]$ is the k-algebra over $\mathbb{N}^{(I)}$. Another example is the product of two rings of polynomials $k[X] \times k[Y]$ is an algebra over a monoid.

The notion of quasi-monoid k-algebras is defined as follows and generalizes the notion introduced by Heindorf [H1] and Evans [E] in the context of Boolean algebras. For, let A be a unitary k-algebra. Suppose that A has a basis $\widehat{M}$ such that $\widehat{M} \cup \{0^A\}$ is a multiplicative unitary monoid and $1^A = 1^{\widehat{M}} \in \widehat{M}$ .First, remark that $\widehat{M}$ satisfies the following property:

$(\ddagger)$ *for every* $s, t \in \widehat{M}$ *: if* $st \neq 0^A$ *then* $st \in \widehat{M}$

Notice that the following universal property holds too:

Proposition 1.3. *Let C be a k-algebra and $f: \widehat{M} \to C$ be a function satisfying $f(0^A) = 0^C$, $f(1^{\widehat{M}}) = 1^C$ and for any $s, t \in \widehat{M}$ if $st \in \widehat{M}$ then $f(s)f(t) = f(st)$. Then f is extendable in a unique algebra homomorphism $\hat{f}: A \to C$.*

The algebra A is called the *quasi-monoid k-algebra* (over $\widehat{M}$) and A is denoted by $[\widehat{M}]$. Hence,

Claim 1.4. *Every monoid algebra is a quasi-monoid algebra.*

The converse is not true, to see this consider $A := k[X] / (X^2)$. Then A is a quasi-monoid algebra that is not a monoid-algebra over k. Indeed $A = k[\widehat{N}]$] where $\widehat{N} = \{1, X\}$ and $X^2 = 0$ in A. But there is no monoid M such that $A = k[M]$. Otherwise if such a M exists, then$1^{k[M]} = 1^M \in M$ and thus M contains only another one element T . Hence $T^2 = 1$ or $T^2 = T$ 1, with $T = \alpha + \beta X$. This implies that either $T = 0$ or $T = \pm 1$. A contradiction.

II. FREE *K*-ALGEBRAS OVER A MONOID OR A QUASI-MONOID

We begin by stating well-known results2.1and2.2on monoid algebras.

Proposition 2.1.

(a) Let M be a monoid, A be a k-algebra and $f: M \to A$ be a function satisfying $f(1^M) = 1^A$ $f(s)f(t) = f(st)$ and for any $s, t \in M$ -Then f is extendable in a unique algebra homomorphism $\hat{f}: k[M] \to A$.

(b) Let M and N be two monoids and $f: M \to N$ be a monoid's homomorphism such that $f(1^M) = 1^N$ Then * is extendable in a unique algebra homomorphism $\hat{f}: k[M] \to k[N]$ satisfying $\hat{f}(st) = f(st) = f(s)f(t) = \hat{f}(s)\hat{f}(t)$ and $\hat{f}(1^M) = 1^N$. Moreover ;

i. If f is one-to-one then $\hat{f}$ is one-to-one, and

ii. If f onto then $\hat{f}$ is onto

Let $\langle M_i : i \in I \rangle$ be a family of monoids. Then the product monoid $M := \prod_{i\in I} M_i$ exists with the operation $\langle x^i \rangle_i \cdot^M \langle y^i \rangle_i = \langle x^i \cdot^{M_i} y^i \rangle_i$ and $1^M = \langle 1^{M_i} \rangle_i$. Then the weak product $\prod_{i\in I}^{w} M_i$ d enotes the submonoid of M consisting of $x = \langle x^i \rangle_i$ satisfying is $\sigma(x) := \{i \in I : x^i \neq 1^{M_i}\}$ is finite. Let $i_l: M_l \to M$ be the canonical monoid embedding defined by $x^l = t$ and $x^i = 1^{M_i}$ for $i \neq l$ for $t \in M_l$. By proposition **2.1**, i_l is extendable in a unique algebra homomorphism $\hat{i_l}: k[M_l] \to k[\prod_{i\in I}^{w} M_i]$. The following fact shows that $k[\prod_{i\in I}^{w} M_i]$ is the tensor product (also called "free product") of the family $\langle k[M_i] : i \in I \rangle$. [J, §3.7], Also Part (b) follows from Part (a).

Proposition 2.2.

(a) Let $\langle M_i : i \in I \rangle$ be a family of monoids. For every commutative k-algebra D and for any sequence of k-algebra homomorphism $f_l: k[M_l] \to D$ such that $f_l(1^{k[M_l]}) = 1^D$ for $l \in I$, there is a unique k-algebra homomorphism $\hat{f}: k[\prod_{i\in I}^{w} M_i] \longrightarrow D$ such that $f_l = \hat{f} \circ \hat{i_l}$ for $\in I$. Hence $k[\prod_{i\in I}^{w} M_i]$ is the tensor product $\otimes_{i\in I}$ $k[M_i]$ of the family $\langle k[M_i] : i \in I \rangle$ in the category of k-algebras.

(b) Any k-algebra is an algebra homomorphic image of $k[M]$ for some monoid M.

To prove that the class of algebras over monoids is closed under finite product, we extend the "lexicographic sum of posets indexed by a poset" in the following way. Let $(L, \wedge)$ be a semilattice, and let $\langle M_i : i \in L \rangle$ be a sequence of monoids. We assume that the M_i 's are pairwise disjoint. We define a monoid $\sum_{i\in L}^{\oplus} M_i$.The universe of $\sum_{i\in L}^{\oplus} M_i$ is $M := \bigcup_{i\in L} M_i$. We shall define an operation $\cdot^M$ on M. For $i \in L$, let $g_{i,i}: M_i \to M_i$ be the identity and for $i < j \in L$ let $g_{i,j}: M_j \to M_i$ be defined as $g_{i,j}(u) = 1^{M_i}$for any $u \in M_j$ so for $i \leq j \leq k$ we have $g_{i,k} = g_{i,j} \circ g_{j,k}$ and each $g_{i,j}$ is a monoid homomorphism that preserves the unit. For $s, t \in M$, let $i, j \in L$ be such $s \in M_i$ and $t \in M_j$ and we set

$s \cdot^M t := g_{i\wedge j, i}(s) \cdot^{M_{i\wedge j}} g_{i\wedge j, j}(t)$

It is easy to check that $\sum_{i\in L}^{\oplus} M_i$ is a monoid (the unit is1^{M_e} where e is the unit of L). The structure $M := \sum_{i\in L}^{\oplus} M_i$is called the *L-sum of the sequence* $\langle M_i : i \in L \rangle$ of monoids. Notice that the direct sum $\lim \langle M_i, g_{i,j} \rangle$ is trivial: if L has a smallest element 0 then $\lim \langle M_i, g_{i,j} \rangle \cong M_0$ and $\lim \langle M_i, g_{i,j} \rangle = \{1\}$otherwise. Also $\cdot^M$ is defined in a more explicit way as follows. For $s \in M_i$ and $t \in M_j$: We shall

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$$\begin{cases} s.^{M}t = s.^{M_i}t & \text{if } i=j \\ s.^{M}t = s = t.^{M}s & \text{if } i<j \\ s.^{M}t = t = t.^{M}s & \text{if } j<i \\ s.^{M}t = 1^{M_{i\wedge j}} = t.^{M}s & \text{if } i \text{ and } j \text{ are incomparable} \end{cases}$$

We shall construct a homomorphism $\hat{f}: k[\sum_{i\in L}^{\oplus} M_i] \longrightarrow \prod_{i\in L} k[M_i]$. We denote $\sum_{i\in L}^{\oplus} M_i$ by M. Recall that $1^M = 1^{M_e}$ where e is the unit of L. For $i \in L$, let $f_i: M \longrightarrow k[M_i]$ defined as follows:

$$\begin{cases} f_i(u) = u & \text{if } u \in M_i \\ f_i(u) = 1^{k[M_i]} & \text{if } u \in M_j \quad \text{and } j > i \\ f_i(u) = 0^{k[M_i]} & \text{if } u \in M_j \text{ and } j \ngeq i \end{cases}$$

Recall that $1^{M_\ell} = 1^{k[M_\ell]}$ for any $\ell \in L$. Notice that $f_i(1^M) = 1^{k[M_i]}$. Obviously $f_i(vw) = f_i(v)f_i(w)$ for every $v, w \in M$. By proposition 2.1, f_i is extendable in an unique k-algebra homomorphism $f_i: k[M] \longrightarrow k[M_i]$. Let $\hat{f} := \langle f_i \rangle_{i\in L}: k[M] \longrightarrow \prod_{i\in L} k[M_i]$ defined by $\hat{f}(x) := \langle f_i(x) \rangle_{i\in L}$ for $x \in k[M]$. Since each f_i is a homomorphism, $\hat{f}$ is a k-algebra homomorphism. The next lemma will be used to prove that the class of algebras over monoids is closed under finite products and weak product.

Lemma 2.3. Assume that $\langle L, \wedge \rangle$ be a semilattice and $\langle M, i \in L \rangle$ be a sequence of monoids. Then $\hat{f}$ is a k- algebra embedding from $K[\sum_{i\in L}^{\oplus} M_i]$ into $\prod_{i\in L} K[M_i]$.

Proof. Since M is a basis of the k-vector space $K[M]$, for any non-zero $K[M]$,tere are an unique finite subset of σ^x and for each $i \in \sigma^x$ an unique finite subset $\{s_{i,j}^x : j < n^x(i)\} \subseteq M_i$ and an unique finite sequence of non-zeroscalar $\langle \lambda_{i,j}^x \in k : j < n(i) \rangle$ such that $x = \sum_{i\in\sigma^x} \sum_{i<n^x(i)} \lambda_{i,j}^x s_{i,j}^x$.

Let $x \in K[M]$ be such that $\hat{f}(x) = 0$ that is $\hat{f}_i(x) = 0$ for every $i \in L$. By contradiction, suppose that $x \neq 0$. Let l be a maximal element of σ^x and $\tau = \sigma^x \setminus \{l\}$. By the definition, we have

$$x = \sum_{j<n^x(l)} \sum_{i<n^x(i)} \lambda_{i,j}^x s_{i,j}^x + \sum_{i\in\tau} \sum_{j<n^x(i)} \lambda_{i,j}^x s_{i,j}^x.$$

and $\hat{f}(x) = 0$. From the choice of l and the definition of $\hat{f}_l$, it follows that:

$$0^{K[M_l]} = \hat{f}(x) = \sum_{j<n^x(l)} \sum_{i<n^x(i)} \lambda_{l,j}^x s_{l,j}^x$$

Since the $s_{l,j}^x$'s are k-linearly independent, $\lambda_{l,j}^x = 0$ for ever $j < n^x(l)$. A contradiction. So $x = 0$. Cqfd

Let $\langle A_i : i \in I \rangle$ be a sequence of unitary k-algebras. For $x := \langle x_i \rangle_{i\in I} \in A := \prod_i A_i$ we set $\sigma^0(x) = \{i \in I : x_i \neq 0\}$. We denote by $\prod_{i\in I}^{W} A_i$ the subalgebra of A generated by $\{1^A\} \cup \{x \in A : \sigma^0(x) \text{ is finite }\}$. Hence $\prod_{i\in I}^{W} A_i$ is a (unitary) k-algebra and $\langle A_i : i \in I \rangle$ is called the weak product of the family . Notice that $x := \langle x_i \rangle_{i\in I} \in A := \prod_i A_i$ if there is such that $\{i \in I : x_i \neq 0 \text{ or } x_i \neq \lambda 1^{M_i}\}$ is finite. If $\langle A_i : i \in I \rangle$ is a family of monoid k-algebras, then so is for the weak product $\prod_{i\in I}^{W} A_i$ (see corollary 2.6).

Let L be a semilattice with a unit $e := 1^L$ and let $\langle M_i : i \in L \rangle$ be a sequence of monoids. We set $L^- := L \setminus \{e\}, \overrightarrow{M}^- := \langle M_i : i \in L^- \rangle$ and $A^- := \prod_{i\in L^-} k[M_i]$. Since $k[M_i]$ has a unit 1^{M_i} for any $i \in L^-$, the algebra A^- has a unit $1^{A^-} := \langle 1^{M_i} \rangle_{i\in L^-}$. We denote by $k[\overrightarrow{M}^-, M_e]$ the subalgebras of $A^- \times k[M_e]$ generated by $G^- \cup G^e$ where

$G^- := \{\langle x, 0 \rangle \in A^- \times k[M_e] : \mathrm{supp}^0(x) \text{is finite}\}$, and

$G^e := \{\langle \lambda 1^{A^-}, 0 \rangle \in A^- \times k[M_e] : \lambda \in k \text{ and } s \in M_e\}$.

Also we identify the algabras $A^- \times k[M_e]$ and $\prod_{i\in L} k[M_i]$. So

$$k[\overrightarrow{M}^-, M_e] \subseteq A^- \times k[M_e] := (\prod_{i\in L^-} k[M_i] \times k[M_e] = \prod_{i\in L} k[M_i]$$

Finally for a poset P we denote by $\mathrm{Max}(P)$ the set of maximal elements of P

Lemma 2.4.Assume that $\langle L, \wedge \rangle$be a semilattice with a unit e satisfies : (*) the set $L \setminus \mathrm{Max}(L \setminus \{e\})$ is finite.

Let $\langle M_i : i \in L \rangle$ be a sequence of monoidsindexed by $\langle L, \wedge \rangle$. . The k-algebra homomrphism $\hat{f}$ is an isomorphism from $K[\sum_{i\in L}^{\oplus} M_i]$ into $k[\overrightarrow{M}^-, M_e]$.

Proof. Note that (*) implies

(†) every chain of L is finite, and

(‡) for every $i \in L$: if $i \neq e := 1^L$ then $\{j \in L : j \leq i\}$ is finite.

By Lemma 2.3 $\hat{f}: K[\sum_{i\in L}^{\oplus} M_i] \longrightarrow \prod_{i\in L} K[M_i]$ is an embedding. We show that $k[\overrightarrow{M}^-, M_e] = \mathrm{Rng}(\hat{f})$. We first prove that $\mathrm{Rng}(\hat{f}) \subseteq k[\overrightarrow{M}^-, M_e]$ recall that. Recall that

(P1) for any $\ell \in L$: if $s \in M_\ell$ then $\hat{f}_\ell(s) = s$, $\hat{f}_i(s) = 1$ for $i < \ell$ and $\hat{f}_i(s) = 0$ for $i \nleq \ell$.

Let $M^- = \cup \overrightarrow{M}^-$, properties (*) and (‡) of L and (P1) implies that $\hat{f}(s) \epsilon G^-$ for any $s \in M^-$. Hence:

(P2) $\hat{f}(M^-) \subseteq G^- \sqsubseteq k[\overrightarrow{M}^-, M_e]$.

For each $\ell \in L$ and $s \in M_\ell$ we set $\bar{s} = \langle s_i \rangle_{i\in L} \in \prod_{i\in L} K[M_i]$ where $s_\ell = s$ and $s_i = 0$ otherwise. We show the following fact:

(**) For every $\ell \in L^-$and $s \in M_\ell$ there is $x_s \in k[\overrightarrow{M}^-, M_e]$ such that $\hat{f}(x_s) = \bar{s}$.

By contradiction, suppose that for some $\ell \in L^-$ and $s \in M_\ell$, ther is no $x_s \in k[\overrightarrow{M}^-, M_e]$ such that $\hat{f}(x_s) = \bar{s}$. Let $\sigma^{<\ell} = \{i \in L : i < \ell\}$. Since $\ell \in L^-$, by (‡),$\sigma^{<\ell}$ is finite. By (*), we may assume that ℓ is minimal with respect to this property. Let $i \in \sigma^{<\ell}$. We choose $t_i \in M_i$. Hence $\hat{f}_\ell(t_i) = 1^{M_\ell}$, and by minimality of ℓ , let $x^i \in k[M]$ such that $\hat{f}(x^i) = 1^{M_i}$. Let $y = \sum_{i\in\sigma^{<\ell}} x^i$ .So $y \in k[\overrightarrow{M}^-, M_e]$, and $\hat{f}_i(y) = 1^{K[M_i]}$ for any $i < \ell$ and $\hat{f}_i(y) = 0^{K[M_i]}$ otherwise. Let $x = s - y$. We have $x \in k[\overrightarrow{M}^-, M_e]$, $\hat{f}_\ell(x) = s$ and $\hat{f}_i(x) = 0^{K[M_i]}$ for $i \neq \ell$. So $x_s := x$ is as required in (**).

Let $s \in M_e$. By (P1), $\hat{f}_e(s) = s$ and $\hat{f}_i(s) = 1^{K[M_i]}$ for $i \neq e$. By (*),$R := L \setminus Max(L \setminus \{e\})$ is finite. By (**), for each $\ell \in R$ there is $y_\ell \in k[\overrightarrow{M}^-, M_e]$ such that $\hat{f}(y_\ell) = 1^{M_\ell}$ .Let $y = \sum_{\ell\in R} y_\ell$ and $z = s - y$. Then $y, z \in k[\overrightarrow{M}^-, M_e]$ and $\hat{f}(z) = \langle 1^{A^-}, s \rangle \in G^e \sqsubseteq k[\overrightarrow{M}^-, M_e]$. Since $s = y + z$, $\hat{f}(s) \in k[\overrightarrow{M}^-, M_e]$and thus

(P3) $\hat{f}(M_e) \sqsubseteq k[\overrightarrow{M}^-, M_e]$.

Since M generates the algebra $k[M]$, it follows from (P2) and (P3) that:

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(P4) $Rng(\hat{f}) \subseteq k[\vec{M}^-, M_e]$

Next we prove $k[\vec{M}^-, M_e] \subseteq Rng(\hat{f})$. We have seen that for every $s \in M_e$ there is $z \in k[\vec{M}^-, M_e]$ such that $\hat{f}(z) = \langle 1^{A^-}, s\rangle \in G^e$. Hence:

(P5) $G^e \subseteq Rng(\hat{f})$.

Let B be the subalgebra of $\prod_{i\in L} K[M_i]$ generated by $\{\bar{s} : s \in M^-\}$. Trivially $G^- \subseteq B$. By (**), for any $s \in M^-$ there is $x_s \in k[\vec{M}^-, M_e]$ such that $\hat{f}(x_s) = \bar{s}$. So

(P6) $G^- \subseteq B \subseteq Rng(\hat{f})$.

Now $k[\vec{M}^-, M_e] \subseteq Rng(\hat{f})$ follows from (P5) and (P6). So $\hat{f}$ is a k-algebraic isomorphism between $k[M]$ and $\prod^{w}_{i\in L} K[M_i]$.

We apply the above result to the products of algebras.

Corrollary 2.5. Let L be a finite semilattice and $\langle M_i : i \in L\rangle$ be a sequence of monoids. Then the algebras $K[\sum^{\oplus}_{i\in L} M_i]$ and $\prod_{i\in L} K[M_i]$ are isomorphic.

Proof. Let *e* be the unit of *L*. It is obvious that $k[\vec{M}^-, M_e] = \prod_{i\in L} K[M_i]$. Now the result follows from Lemma 2.4.

We extend Corollary 2.5. for weak product of k-algebras over monoids. The proof uses the first part of 2.4.

Corrollary 2.6. Let $\langle A_i : i \in I\rangle$ be a sequence of unitary k-algebras over monoids. Then the weak product $\prod^{w}_{i\in I} A_i$ is a unitary k-algebra over a monoid.

Proof. First if *I* is finite, the result follows from proposition 2.5. So we may assume that I is infinite. We choose $i_0 \in I$ and we add a new element i^+ to I. Let $I^+ = I \cup \{i^+\}$. Let $<$ be the order relation defined on I^+ by $i_0 < i < i^+$ for every $i \in I \setminus \{i_0\}$. So i and i' are incomparable for distinct $i, i' \in I \setminus \{i_0\}$, and I^+ is a semilattice with i^+ as unit. For each $i \in I$, let M_i be a monoid such that $A_i = K[M_i]$. Also let $M_{i^+}:=\{e\}$. Notice that $k[M_{i^+}] = ke$. We set $M := \sum^{\oplus}_{i\in L} M_i$ and $M^+ := \sum^{\oplus}_{i\in L} M_{i^+}$. So M is a (non-unitary) monoid and $M^+ = M \cup \{e\}$ is a monoid with e as unit.

Let $\vec{M} := \langle M_i : i \in L\rangle$. It suffices to show that $\prod^{w}_{i\in L} K[M_i]$ and $k[\vec{M}^-, M_{i^+}]$ are isomorphic. Let $g: k[\vec{M}^-, M_{i^+}] \longrightarrow A$ defined by $g(\langle x, \lambda e\rangle) = x + \lambda 1^A$. Since $G^- \cap G^e = \{0\}$, g is a k-algebra monomorphism. So it suffices to verify that $Rng(g) = \prod^{w}_{i\in I} A_i$, but this is trivial.

Example 2.7. (1) Let $L = 2 := \{0, 1\}$ with $0 < 1$. That is $\langle 2, .\rangle$ is the semilattice where . is the usual multiplication. Let M_0 and M_1 be unitary monoids, $M = \sum^{\oplus}_{i\in 2} M_i$ and $\hat{f}$ be the isomorphism from $k[M]$ onto $k[M_0] \times k[M_1]$ defined in the proof of Proposition 2.5 Then $k[M]$ for $u \in M_0$ and $\hat{f}(v) = \langle 1^{M_0}, v\rangle$ for $\in M_1$. Also notice that $1^M = 1^{M_1}$ and that $\hat{f}(1^M) = \langle 1^{M_0}, 1^{M_1}\rangle$.

In particular the product of rings of polynomials $k[X] \times k[Y]$ is an algebra over a monoid.

(2) From Corollary 2.5, it follows that for a finite semilattice L and a sequence of monoids $\langle M_i : i \in L\rangle$, we have $k[\sum^{\oplus}_{i\in L} M_i] \cong k[\sum^{\oplus}_{j<n} M_j]$ where $n = |L|$ and $n := \{0, 1, \dots, n-1\}$ is considered as a semilattice (with $j.l = j$ iff $j \leq l$).

In particular, for any field k and any finite semilattice L, the algebras $k[L]$ and k^n are isomorphic, where $n = |L|$. Indeed, for $i < n$ let $M_i = \{1^{M_i}\}$ Then $L \cong \sum^{\oplus}_{i\in L} M_i$, and by Corollary 2.5 and thus $k[L] \cong k[\sum^{\oplus}_{i\in L} M_i] \cong k[\sum^{\oplus}_{j<n} M_j] \cong \prod_{j<n} k[M_j] \cong k^n$. semilatice).

(3) Characterize the field k and the semilattice L such that the weak product $\prod^{w}_{i\in L} A_i$ of family $\langle A_i : i \in L\rangle$.of unitary k-algebras over monoids is also a k-algebra over a monoid.

3 Free k-algabras over a quasi-mnoid

We recall that a quasi-monoid k-algebra $k[\hat{M}]$ is an algebra such that $\hat{M}$ is k-basis that satisfy :

(‡) for every $s, t \in \hat{M}$: if $st \neq 0$ then $st \in \hat{M}$ (trivially $0 \notin \hat{M}$).

Let $D^{\hat{M}} := \{\langle u, v\rangle \in \hat{M}^2 : uv \neq 0\}$. So . : $D^{\hat{M}} \to \hat{M}$ is a partial operation on $\hat{M}$ satisfying the following properties.

(Q_0) For any $\langle u, v\rangle \in D^{\hat{M}}$, $u.v \in \hat{M}$.

(Q_1) For any $\langle u, v\rangle \in D^{\hat{M}}$: $\langle v, u\rangle \in D^{\hat{M}}$ and $u.v = v.u$

(Q_2) There is $1 \in \hat{M}$ such that $\langle 1, u\rangle \in D^{\hat{M}}$ and $1.u = u$ for any $\in \hat{M}$.

(Q_3) For any $, v, w \in \hat{M}$; if $\langle v, w\rangle \in D^{\hat{M}}$ and $\langle u, v.w\rangle \in D^{\hat{M}}$ then $\langle u, v\rangle \in D^{\hat{M}}$, $\langle u.v, w\rangle \in D^{\hat{M}}$ and $u.(v.w) = (u.v).w$.

A structure $\langle \hat{M}, .\rangle$ satisfying $(Q_0) - (Q_3)$ is called a quasi-monoid . We denote $\cdot$ by $\cdot^{\hat{M}}$. We define the k-algebra A over $\hat{M}$ as follows. The set $\hat{M}$ is a basis of the k-vector space A. Next we define the multiplication $\cdot^A$ on $\hat{M}$ by $u \cdot^A v = u \cdot^{\hat{M}} v$ if $\langle u, v\rangle \in D^{\hat{M}}$ and $u \cdot^A v = 0^A$ otherwise. By bilinearity, $\cdot^A$ is is extendable in multiplication on A. Hence A is quasi-monoid algebra and $A = k[\hat{M}]$.

The universal propriety of algebras over quasi-monoids, corresponding to 2.1, is the following.

Claim 3.1. Let $\hat{M}$ be a quasi-monoid. A be a k-algebra and So . : $f: \hat{M} \to A$ be a function satisfyingf$(0^M) = 0^A$, $f(1^{\hat{M}}) = 1^A$ and for any $s, t \in \hat{M}$: : if $st \in \hat{M}$ then $f(s)f(t) = f(st)$ .Then f is extendable in a unique algebra homomorphism $\hat{f}: k[\hat{M}] \to A$.

We define $\prod^{w}_{i\in I} M_i$ and $\sum^{\oplus}_{i\in L} M_i$ for quasi-monoids , as it was done for monoids . Let $\langle M_i : i \in I\rangle$ be a family of quasi-monoids. Then the product quasi-monoid $M := \prod_{i\in I} M_i$ exists with the operation $\langle x^i\rangle_i .^M \langle y_i\rangle_i = \langle x^i .^{M_i} y^i\rangle_i$ (ifx$^i .^{M_i} y^i$ exists for every i) and $1^M = \langle 1^{M_i}\rangle_i$. Then weak product $\prod^{w}_{i\in I} M_i$ denotes the sub-quasi-monoids of M consisting of $x = \langle x^i\rangle_i$ satisfying $\sigma(x) := \{i \in I : x^i \neq 1^{M_i}\}$ is finite .

Let L be a monoid and $\langle M_i : i \in L\rangle$ be a sequence of quasi-monoids. We asumme that the M_i 's are pairwise disjoint . We denote by $\sum^{\oplus}_{i\in L} M_i$ the following structure : the universe of $\sum^{\oplus}_{i\in L} M_i$ is $M = \bigcup_{i\in L} M_i$, and the operation $.^M$ on $\sum^{\oplus}_{i\in L} M_i$ is defined as follows . For $s \in M_i$ and $t \in M_j$ with $, j \in L$, we set :

$$\begin{cases} s.^M t = s.^{M_i} t & \text{if } i = j \text{ and } s.^{M_i} t \text{ exists,} \\ s.^M t = t.^M s & \text{if } i.j = i \text{ and } i \neq j, \\ s.^M t = t = t.^M s & \text{if } i.j = j \text{ and } i \neq j, \\ s.^M t = 1^{M_k} = t.^M s & \text{if } i.j := k \text{ and } i \neq j \neq k \neq i. \end{cases}$$

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The structure $M := \sum_{i\in L}^{\oplus} M_i$ is called the L-sum of sequence $\langle M_i : i \in L\rangle$ of quasi-monoids . It is easy to check that M is a quasi-monoid.

The proof of the next result is left to the reader .

Proposition 3.2. Let $\langle \widehat{M}_i : i \in I\rangle$ be a family of quasi-monoids. For every commutative k-algebra D and for any sequence of k-algebra homorphisms $f_l : k[\widehat{M}_l] \to D$ such that $f_l(0^{k[\widehat{M}_l]}) = 0^D$ for $l \in I$ and $f_l(1^{k[\widehat{M}_l]}) = 1^D$ for $l \in I$, there is a unique k-algebra homomorphism $\hat{f} : k[\widehat{M}_{i\in I}^{w}] \to D$ such that $f_l = \hat{f} \circ \hat{\imath}_l$ for $l \in I$.($\hat{\imath}_l$ is defined after proposition 2.1.)

III. APPLICATIONS

4.0 Upper-lattice algebras

First we recall the notion of upper-lattice algebra developed by Bekkali and Zhani in [BZ1, BZ2] and after we shall apply the results of section 2 to that class that is exactly the class of algebras over semi groups (semi-group algebras). Moreover, our work implies the same results to free poset algebras P(P) (which are upper semi lattices by theorem 4.9).

Let T be an upper semi-lattice i.e. every pair {p,q} of members of T has a least upper bound p∨q. Let T be an upper-semi lattice with a first element 0^T. For $t \in T$, we set $b_t = [t, \to) := \{u \in T : u \geq t\}$. Let $G_T = \{b_t \in \mathcal{P}(T) : t \in T\}$. The subalgebra $B(T)$ of $\mathcal{P}(T)$ is called *the upper-lattice algebra* over T. Notice that G_T is a unitary semi lattice generated $B(T)$ and $1^{B(T)} = b_{0^T}$.

Proposition 4. 1. Let B be a Boolean algebra. The following properties are equivalent.

(1) B is a Boolean algebra over a unitary semilattice.

(2) B is an upper-lattice algebra.

Proof. $(1) \Rightarrow (2)$. Suppose that $B = \mathbb{F}_2[N]$. For $u, v \in N$, set $u \leq v$ if $uv = v$. So $\langle N, \leq\rangle$ is a unitary semilattice $(u \wedge v := uv)$ with a last element 1^N.Let $\langle T, \leq\rangle = \langle N, \geq\rangle$ is a join unitary semilattice $(u \vee^T v := uv)$ with a least element 1^N. For $t \in T$, set $b_t = [t, \to) := \{u \in T : u \geq^T t\}$, let $G_T = \{b_t \in \mathcal{P}(T) : t \in T\}$. Notice that $B(T)$ is an upper –lattice algebra. It is easy to check that $x \longmapsto b_t$ from M onto G_T is extendable in an isomorphism from $\mathbb{F}_2[N]$to $B(T)$.

$(2) \Rightarrow (1)$. Let $B(T)$ be an upper-lattice algebra. Recall that G_T is an unitary semilattice generating $0^{B(T)} \notin G_T$, and that $1^{B(T)} = T = b_{0^T}$. So it suffices to prove that G_T is a linear independent set over $\mathbb{F}_2$. This is a consequence of the following general result, that will be also applied in Proposition 5.1, for completeness, we recall the proof of this fact.

Fact 4.2 Let S be a poset and A be a subalgebra of $\mathcal{P}(S)$. So A is a $\mathbb{F}_2$-ring. Let $\langle a_i : i < m\rangle$be a finite sequence of pairwise distinct non-zero elements of A. Suppose that for every $i < m$, a_i has a first element $s_i := \min(a_i)$, and if $i < j < m$ then $s_i \neq s_j$. Then $\langle a_i : i < m\rangle$ are $\mathbb{F}_2$-independent.

Proof. We prove this fact by induction on m. If m=0 then $a_0 \neq 0$. Next, suppose that the lemma holds for $m-1$. Suppose that $\sum_{i<m} a_i$. We shall find a contradiction. Recall that $a + b = 0$ iff $a = b$. We may assume that $s_{m-1}$is a minimal element of $\{s_i : i < m\}$. We have $a := a_{m-1=} \sum_{i<m-1} a_i$ and (*) : $s := \min(a) = s_{m-1} \ngeq s_i$ for $i < m-1$. Since $s \epsilon a$, let $j < m-1$ such that $s \epsilon a_j$ .Since $s_j := \min(a_j)$, $s_j \leq s$. Since $j < m-1$, by hypothesis, $s_j \neq s$, and thus $s_j < s$ that contradicts (*). We have proved the fact.

A Boolean algebra B is called a *Boolean algebra over a unitary semilattice* M if B is isomorphic to $\mathbb{F}_2[M]$ where M is a monoid. Notice that such a M must be unitary semilattice (since $x^2 = x$). As a consequence of the above section2, we have the following result on Boolean algebras.

Proposition 4. 3.

(1) Let M be an unitary semi-lattice, A be a Boolean algebra and $f : M \longrightarrow A$ be such that $f(1^M) = 1^A$ and $f(st) = f(s)f(t)$ for $s, t \epsilon M$. Then f is extendable in a Boolean homomorphism from $\mathbb{F}_2[M]$ into A.

(2) Let $\langle M_i : i \in I\rangle$ be a family of unitary semilattices. Then

(a) $\prod_{i\in I} M_i$ and $\prod_{i\in I}^{W} M_i$ are unitary semilattices.

(b) The tensor product (that is called the free product) $\otimes_{i\in I} \mathbb{F}_2[M]$ is the Boolean algebra over the unitary semilattice$\prod_{i\in I}^{W} M_i$.

(3) The class of algebras over unitary semi-lattices is closed under weak product, and thus under finite products.

(4) Any Boolean algebra is a homomorphic image of an algebra over a unitary semi-lattice.

Proof. (1) is an application of Proposition 2.1. (2a) is easy and (2b) follows from Proposition 2.2(a). (3) is a special case of Propositions 2.5 and 2.6. (4) follows from the proof of Proposition 2.2(b), considering the monoid $M_a = \{1^{M_a}, a\}$ with $a^2 = a$ for any member a of M_a.

Remark4.4. In [BZ1, theorem 3.10], actually, there is a normal form of the fact that an upper-lattice algebra has a basis of non zero elements using symmetric difference, Δ. The same normal form holds in semi group algebra and free poset algebras, see [BZ2]. Moreover, notice that algebras over semigroups or semigroup algebras, as we usually call them, have a neat characterization by theorem 4.9. as well as free poset algebras by theorem 4.8. For the complete proofs and discussions of these theorems, see [BZ1, BZ2]. In this paper, theorems 4.4, 4.5, 4.7 and 4.10 are the connection between upper semi-lattices, semi group algebras and free poset algebras by their different representations showing how actually these algebras are built up from inside.

Notice that all the above results (1)-(4) hold when we consider a commutative field *k* of characteristic 2 instead of $\mathbb{F}_2$.

Theorem 4.5. The following statements are equivalent for any Boolean algebra B.

1. B is isomorphic to an upper semi-lattice algebra.
2. B is generated by $H \subseteq B$ so that $0 \notin H$, H is disjunctive, containing 1 and closed under multiplication.

Next theorem characterizes $\mathcal{I}d(T)$ (the set of ideals of T), for any upper semi-lattice $(T, \leq)$, see [BZ1]. Let B be a Boolean algebra and set $\mathcal{U}lt(B)$ its stone space.

Theorem 4.6. The following statements are equivalent.

1. B is isomorphic to $B(T)$, where $\langle T, \leq\rangle$ is an upper semi-lattice, with least element.

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2. X is homeomorphic to $\mathfrak{Id}(T)$, the set of ideals of an upper semi-lattice T with a least element, endowed with Tychonoff's topology inherited from 2^T.
3. X is homeomorphic to $\mathcal{F}(S)$, the set of filters over S, where S is a unitary semi-lattice, endowed with Tychonoff's topology inherited from 2^S.
4. There is a multiplication '**.**' on X so that $(X,.)$ is a unitary semi-lattice and '.' Is a continous mapping on $X \times X$ (i.e. $(x, y) \longmapsto x.y$ is continuous).
5. At this stage notice that algebras over semigroup, have a neat characterization by theorem 4.9. as well as free poset algebras by theorem 4.8. For the complete proofs and discussions of these theorems, see [BZ1, BZ2].

5.1 Free poset algebras

Let $(P, \leq)$ be a poset. A *free poset algebra over P* is a Boolean algebra *A* having P as set of generators such that:

(1) $p \leq_P q$ implies $p \leq_A q$,

(2) For every BA *B* and every mapping $f: P \to B$, if $p \leq_P q$ implies that $f(p) \leq f(q)$, then there is a homomorphism $g: A \to B$ such that $g \restriction P = f$. Note that for P a poset consisting of isolated elements, that is with $\leq$ the identity, a free poset algebra is just a free Boolean algebra over P. In particular, if P is infinite, then this algebra is atomless.

For any poset P, a final segment of P is a subset M of P such that if $p \in M$ and $p \leq q$, then also q$\in M$. For any poset P, $Fs(P)$ is the collection of all final segment of P. Note that $\emptyset$ is a final segment of P.

Proposotion 4.7. Suppose that P is a poset, then $Fs(P)$ is a closed subspace of P. Moreover, $clop(Fs(P))$ is a free poset algebra on P.

Proposition 4.8. Every poset algebra is a semigroup algebra.

Let P be a psoet. An antichain in P is a collection of pairwise incomparable elements of P. $Ant\,(P)$is the set of all finite antichains of P. Note that $\emptyset \in Ant\,(P)$, and{p} $Ant\,(P)$ for all $p \in P$. We define a relation $\preccurlyeq$ on $Ant\,(P)$ by:

$$\sigma \preccurlyeq \tau \leftrightarrow \forall p \in \sigma \exists q \in \tau\ (q \leq p)$$

Theorem 4.9. Let P be a poset, Then:

1. $(Ant(P), \preccurlyeq)$ is an upper semi-lattice.
2. For each $p \in P$ let $f(p) = \{p\}$. Then f is an order anti-isomorphism of P into $Ant\,(P)$.
3. $Tail(Ant(P))$ is isomorphic t o a free poset algebra over P.

4.2.Semi-group algebras

The following theorem gives a concrete construction of semi-group algebras. For, let $(M,\wedge)$ be an idempotent semi-lattice with 0 and1, and let *A* be the free Boolean algebra with generators x_p for $p \in M$, and let *I* be the ideal generated by the set

$\{(x_p, x_q)\Delta x_{p\wedge q}: p, q \in M\}$

Theorem 4.10. (D. Monk).

i. $A\,/\,I$ is a semi-group algebra;

ii. Every semi-group algebra is isomorphic to some $A\,/\,I$ as above;

iii. The Stone space $\mathfrak{Ult}(A\,/\,I)$ is homomorphic to $\mathcal{F}(S)$, where $(S,\wedge)$ is a unitary meet semi-lattice.

Corrollary 4.11. Every semi-group algebra is isomorphic to an upper semi-lattice algebra as well as any pseudo-tree algebra with a single root.

Proof. By (iii.) in Theorem 4.4. we have the first part of the statement; now if $(T, <)$ is a pseudo tree, then $H := \{b_t$: t$\in T\} \cup \{0\}$generates $B(T)$. Notice, definition 1.2, that $B(T)$ is a semigroup algebra and thus it is an upper semi-lattice by Theorem 4.9.

5 Algebras over unitary quasi-semilattices: quasi upper-lattice Boolean algebra

If $k = \mathbb{F}_2$ and if the unitary quasi-semilattice H, then $\mathbb{F}_2[\hat{H}]$ is called an algebra over an unitary semilattice or semi-ring algebra. The study of semi-ring algebras was initied by Heindorf [H1, H2] and Evan [E]. We define the notion of quasi-upper-lattice algebra. A poset T is a quasi-upper-semilattice if

(†) for every $p, q \in P$,if $\{p, q\}$ has an upper bound then $\{p, q\}$ has a least upper bound $p \vee q$

Let T be a quasi-upper-semilattice with a first element 0^T. For $t \in T$, let $b_t = [t, \to) := \{u \in T: u \geq^T t\}$. Let $G_T = \{b_t \in \mathcal{P}(T): t \in T\}$ The subalgebra $B(T)$ of $\mathcal{P}(T)$ generated by G_T is called the *quasi-upper-lattice algebra over T*. Notice that G_T is an unitary quasi-semilattice generating $B(T)$ and that $1^{B(T)} = T = b_{0^T}$.

Proposition 5.1. Let B be a Boolean algebra. The following properties are equivalents.

i. B is an algebra over a quasi-monoid.

ii. B is a quasi-upper-lattice algebra.

Proof. The proof is similar to that of proposition 4.1. (i) $\Leftrightarrow$ (ii).

Next, we translate Proposition 3.2 in terms of quasi-upper-lattice algebras.

Proposition 5.2. Let $\langle T_i: i \in I \rangle$ be a family of quasi-upper-lattices.

(1)For every commutative quasi-upper-lattice algebra $B(T)$ and for any sequence of Boolean homomorphisms $f_\ell: B(T_\ell) \to B(T)$ such that $f(1^{B(T_\ell)}) = 1^{B(T)}$ and $f(0^{B(T_\ell)}) = 0^{B(T)}$ for $\ell \in I$, there is a unique Boolean homomorphism $\hat{f}: B(\prod_{i \in I}^{W} T_i) \to B(T)$ such that $f_\ell = \hat{f} \circ \hat{\iota}_\ell$ for $\ell \in I$.

(2)The Boolean algebras $B(\prod_{i \in I}^{W} T_i)$ and $\otimes_{i \in I} B(M_i)$ are (canonically) isomorphic.

(3)The class of Boolean algebras over quasi-upper-lattices is closed under finite product.

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