Practical Implementation of Newton's Method Tested on Quadratic Functions

Awatif M. A. El Siddieg

***Abstract:** In this work we give a detailed look to the Practical implementation of Newton's method tested on quadratic functions. Section (1) speak about the theory of optimization problems, introduce definitions and theorems of linear programming problems , definitions and theorems of quadratic programming problems . Section (2) introduce some methods that has a relationship with our method . In section(3) we look at a method for approximating solutions to equations, solving unconstrained optimization problems. The general theory of the problem is described. Section(4) gives practical implementation of Newton's method tested on quadratic functions to test the theoretical results shown in the work. Section (1)*

***Keywords:** - linear programming problems, Practical implementation of Newton's, quadratic functions.*

I. INTRODUCTION

In this section we speak about the theory of optimization problems, introduce definitions and theorems of linear programming problems, definitions and theorems of quadratic programming problems. Linear programming problems:

Definition(1): Optimization might be defined as the science of determining the(best) solution to certain mathematically defined problems , which are often models of physical reality. It involves the study of optimality criteria for problems.

Example (1):

minimize:

$4a + 5b + 6c$

subject to:

$a + b \geq 11$

$a - b \leq 5$

$c - a - b = 0$

$7a \geq 35 - 12b$

$a \geq 0, b \geq 0, c \geq 0$ Subject to

Solution:

To solve this LP we use the equation c-a-b=0 to put $c=a+b \geq 0$ as $a \geq 0$ and $b \geq 0$) and so the LP is reduced to

Example (2) :

minimize

$4a + 5b + 6(a + b) = 10a + 11b$

subject to

$a + b \geq 11$

$a - b \leq 5$

$7a + 12b \geq 35$

$a \geq 0, b \geq 0$

the minimum occurs at $a - b = 5$ and $a + b = 11$

i.e. $a = 8$ and $b = 3$ with $c = a + b = 11$ and the value of the objective function $10a + 11b = 80 + 33 = 113$.

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Example (3):

maximize :

$5x_1 + 6x_2$

subject to :

$x_1 + x_2 \leq 10$

$x_1 - x_2 \geq 3$

$5x_1 + 4x_2 \leq 35$

$x_1 \geq 0$

$x_2 \geq 0$

Solution:

$5x_1 + 4x_2 = 35$ and

$x_1 - x_2 = 3$

Solving simultaneously, rather than by reading values off the graph, we have that

$5(3 + x_2) + 4x_2 = 35$

i.e. $15 + 9x_2 = 35$

i.e. $x_2 = (20/9) = 2.222$ and

$x_1 = 3 + x_2 = (47/9) = 5.222$

The maximum value is $5(47/9) + 6(20/9) = (355/9) = 39.444$

Definition(2): (Unconstrained optimization problems)

The problem takes the form:

minimize $f(\underline{x})$.

Subject to $\underline{x} \in \Re^n$

Where f is a continuous real valued function .

Definition (3) :

A point $\underline{x} \in \Re^n$ is said to be a relative minimum point or a local $\Re^n$ if $\exists$ an $\mathcal{E} > 0$ such that $f(\underline{x}) \geq f(\underline{x}^*)$ $\forall \underline{x} \in \Re^n$ a strict relative minimum point of over $\Re^n$.

Definition (4): (constrained optimization problems)

The general form of a constrained optimization problem the form[9] :

$$\min_{\underline{x} \in \Re^n} f(\underline{x})$$

$$c_i(\underline{x}) = 0\,,\ i = 1,2,\dots\dots,p \qquad (1)$$

$$c_i(\underline{x}) \geq 0, i = p+1\,, \dots\dots, n \qquad (2)$$

Where c_i is the i^{th} constraint function . the constraints $c_i(\underline{x}) = 0$ are termed equality constraints and the set of such . constraints is denoted by (E) and the constraints $c_i(\underline{x}) \geq 0$ are termed inequality constraints denoted by(I).

Proposition (1): (First order necessary condition)

Let S be a subset of $\Re^n$, and let $f \in \not{c}^{(1)}$ be a function on S . If $\underline{x}^*$ is a relative any minimum point of f over $\underline{p}$, then for any $f \in \Re^n$, that is a feasible direction at $\underline{x}^*$, we have $\nabla f(\underline{x}^*)\underline{p}^T \geq 0[$.

Corollary (1) :

Let S be a subset of $\Re^n$, let $f \in \not{c}^{(1)}$, be a function on $\Re^n$. If $\underline{x}^*$ is a relative minimum point of f and if $\underline{x}^*$ is an interior point of $\Re^n$, then

$\nabla f(\underline{x}^*) = \underline{0}$.

Descent directions at a point :

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Consider the Taylor expansion of $f(\underline{x})$ about $\underline{x}'$ up to the first order term

$f\left(\underline{x}' + \alpha \underline{p}\right) = f(\underline{x}') + \alpha \underline{p}^T g\left(x' + \alpha\theta \underline{p}\right)$

$\alpha > 0$.

Where, $0 < \theta < 1$,

Since $f(\underline{x})$ is smooth enough (i.e, all the partial derivatives are continuous), then $\underline{p}^T g(\underline{x})$ is $< 0, \forall \underline{x}$,then sufficiently close to $\underline{x}'$ (by continuity). Thus if α is taken sufficiently small.

$$\underline{p}^T g\left(\underline{x}' + \alpha\theta \underline{p}\right) < 0$$

More precisely $\exists \overline{\alpha} > 0$ such that

$$\underline{p}^T g\left(\underline{x}' + \alpha\theta \underline{p}\right) < 0 \qquad \forall \alpha \in [0, \overline{\alpha}]$$

Thus $f\left(\underline{x}' + \alpha \underline{p}\right) < f(\underline{x}')$ we notice that if $\underline{p}^T g' < 0$, then the value of f decreases (locally if we move in the direction $\underline{p}$) .

Such a direction $\underline{p}$ is called a descent direction at $\underline{x}'$, and it chamcterized . by :

$$\underline{p}^T g' < 0$$

An example of a descent direction at $\underline{x}'$ is $\underline{p} = -g'$since $-g'^T g' < 0$ provided $\underline{g}' \neq \underline{0}$

Definition (5): (Definite and semi definite matrices)

Let C be symmetric matrix we say that C is positive definite if $\underline{x}^T C\underline{x} > 0\ \forall \underline{x} \in \Re^n$,$\underline{x} \neq \underline{0}$, C is called positive semi definite if

$\underline{x}^T C\underline{x} \geq 0$ for $\forall\ \underline{x} \in \Re^n$.

Section (2)

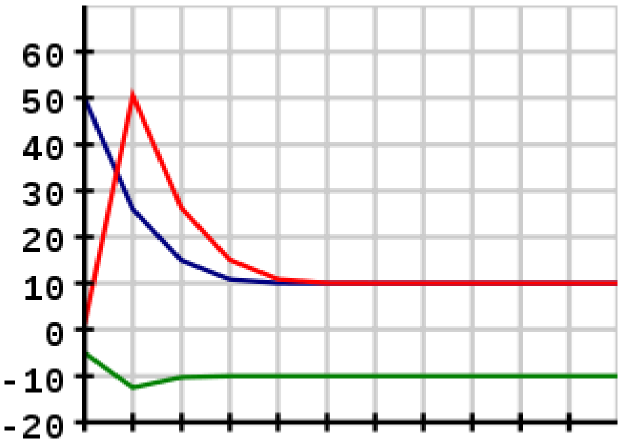
Introduction:

In this section we introduce some methods that has a relationship with our method.

Definition(6):

Quadratic programming (QP) is a special type of mathematical optimization problem. It is the problem of optimizing (minimizing or maximizing) a quadratic function of several variables subject to linear constraints on these variables .

1-Babylonian method:



Graph charting the use of the Babylonian method for approximating the square root of 100 (10) using starting values $x_0 = 50$, $x_0 = 1$, and $x_0 = -5$. Note that using a negative starting value yields the negative root [3].

Perhaps the first algorithm used for approximating $\sqrt{S}$ is known as the "Babylonian method", named after the Babylonians,[1] or "Heron's method", named after the first-century Greek mathematician Hero of Alexandria who gave the first explicit description of the method.[2] It can be derived from (but predates by many centuries) Newton's method. The basic idea is that if x is an overestimate to the square root of a non-negative real number S then S/x will be an underestimate and so the average of these two numbers may reasonably be expected to provide a better approximation (though the formal proof of that assertion depends on the inequality of arithmetic and geometric means that shows this average is always an overestimate of the square root, as noted in the article on square roots, thus assuring convergence). This is a quadratically convergent algorithm, which means that the number of correct digits of the approximation roughly doubles with each iteration. It proceeds as follows:

1. Begin with an arbitrary positive starting value x_0 (the closer to the actual square root of S, the better).
2. Let x_{n+1} be the average of x_n and S / x_n (using the arithmetic mean to approximate the geometric mean).
3. Repeat step 2 until the desired accuracy is achieved.

It can also be represented as:

$$x_0 \approx \sqrt{S}.$$
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right),$$
$$\sqrt{S} = \lim_{n\to\infty} x_n.$$

This algorithm works equally well in the p-adic numbers, but cannot be used to identify real square roots with p-adic square roots; it is easy, for example, to construct a sequence of rational numbers by this method that converges to +3 in the reals, but to −3 in the 2-adics.

Example (1):

Calculate $\sqrt{S}$, where S = 125348, to 6 significant figures, use the rough estimation method above to get x_0. The number of digits in S is $D = 6 = 2\cdot 2 + 2$. So, $n = 2$ and the rough estimate is:

$$x_0 = 6 \cdot 10^2 = 600.000.$$
$$x_1 = \frac{1}{2}\left(x_0 + \frac{S}{x_0}\right) = \frac{1}{2}\left(600.000 + \frac{125348}{600.000}\right) = 404.457.$$
$$x_2 = \frac{1}{2}\left(x_1 + \frac{S}{x_1}\right) = \frac{1}{2}\left(404.457 + \frac{125348}{404.457}\right) = 357.187.$$
$$x_3 = \frac{1}{2}\left(x_2 + \frac{S}{x_2}\right) = \frac{1}{2}\left(357.187 + \frac{125348}{357.187}\right) = 354.059.$$
$$x_4 = \frac{1}{2}\left(x_3 + \frac{S}{x_3}\right) = \frac{1}{2}\left(354.059 + \frac{125348}{354.059}\right) = 354.045.$$
$$x_5 = \frac{1}{2}\left(x_4 + \frac{S}{x_4}\right) = \frac{1}{2}\left(354.045 + \frac{125348}{354.045}\right) = 354.045.$$

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Therefore, $\sqrt{125348} \approx 354.045$.

Convergence:

Let the relative error in x_n be defined by

$$\varepsilon_n = \frac{x_n}{\sqrt{S}} - 1$$

and thus

$$x_n = \sqrt{S} \cdot (1 + \varepsilon_n).$$

Then it can be shown that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2(1+\varepsilon_n)}$$

and thus that

$$0 \le \varepsilon_{n+2} \le \min\left\{\frac{\varepsilon_{n+1}^2}{2}, \frac{\varepsilon_{n+1}}{2}\right\}$$

and consequently that convergence is assured provided that x_0 and S are both positive.

Worst case for convergence:

If using the rough estimate above with the Babylonian method, then the worst cases are:

$$S = 1; \quad x_0 = 2; \quad x_1 = 1.250; \quad \varepsilon_1 = 0.250.$$
$$S = 10; \quad x_0 = 2; \quad x_1 = 3.500; \quad \varepsilon_1 < 0.107.$$
$$S = 10; \quad x_0 = 6; \quad x_1 = 3.833; \quad \varepsilon_1 < 0.213.$$
$$S = 100; \quad x_0 = 6; \quad x_1 = 11.333; \quad \varepsilon_1 < 0.134.$$

Thus in any case,

$$\varepsilon_1 \le 2^{-2}.$$
$$\varepsilon_2 < 2^{-5} < 10^{-1}.$$
$$\varepsilon_3 < 2^{-11} < 10^{-3}.$$
$$\varepsilon_4 < 2^{-23} < 10^{-6}.$$
$$\varepsilon_5 < 2^{-47} < 10^{-14}.$$
$$\varepsilon_6 < 2^{-95} < 10^{-28}.$$
$$\varepsilon_7 < 2^{-191} < 10^{-57}.$$
$$\varepsilon_8 < 2^{-383} < 10^{-115}.$$

Remember that rounding errors will slow the convergence. It is recommended to keep at least one extra digit beyond the desired accuracy of the x_n being calculated to minimize round off error.

II. EXPONENTIAL IDENTITY

Pocket calculators typically implement good routines to compute the exponential function and the natural logarithm, and then compute the square root of S using the identity[*citation needed*]

$$\sqrt{S} = e^{\frac{1}{2}\ln S}.$$

The same identity is used when computing square roots with logarithm tables or slide rules. Method of bisecting intervals:

A simple way to compute a square root is the high/low method, similar to the bisection method. This method involves guessing a number based on known squares, then checking if its square is too high or too low and adjusting accordingly. To find the square root of 20, first note that the square of 5 is 25, and that the square of 4 is 16. As 20 is greater than 16 and less than 25, the square root of 20 must be in between 4 and 5. Guessing 4.5, as the average of 4 and 5, yields 20.25 and is too high. The next step is to guess 4.4, yielding 19.36 and is too low. Therefore, as before, the square root of 20 must be in between 4.4 and 4.5. Continue this pattern until the desired number of decimal places is achieved. For example:

4.45^2 = 19.8025 (too low)
4.47^2 = 19.9809 (too low, but close)
4.48^2 = 20.0704 (too high)
4.475^2 = 20.025625 (too high)
4.473^2 = 20.007729 (too high, but close)
4.472^2 = 19.998784 (too low)

Now it is known that the square root of 20 is between 4.472 and 4.473, so the square root of 20 to the first three decimal places is 4.472.

III. BAKHSHALI APPROXIMATION

This method for finding an approximation to a square root was described in an ancient Indian mathematical manuscript called the Bakhshali manuscript. It is equivalent to two iterations of the Babylonian method beginning with N [2]. The original presentation goes as follows: To calculate $\sqrt{S}$, let N^2 be the nearest perfect square to S. Then, calculate:

$$d = S - N^2$$
$$P = \frac{d}{2N}$$
$$A = N + P$$
$$\sqrt{S} \approx A - \frac{P^2}{2A}$$

This can be also written as:

$$\sqrt{S} \approx N + \frac{d}{2N} - \frac{d^2}{8N^3 + 4Nd} = \frac{8N^4 + 8N^2 d + d^2}{8N^3 + 4Nd} = \frac{N^4 + 6N^2 S + S^2}{4N^3 + 4NS}$$

Example (2):

Find the square root of 152.2756.

Solution :

√ 01 52.27 56 is 1 2. 3 4

01	1*1 ≤ 1 < 2*2	x = 1
01	y = x*x = 1*1 = 1	
00 52	22*2 ≤ 52 < 23*3	x = 2
00 44	y = (20+x)*x = 22*2 = 44	
08 27	243*3 ≤ 827 < 244*4	x = 3
07 29	y = (240+x)*x = 243*3 = 729	
98 56	2464*4 ≤ 9856 < 2465*5	x = 4
98 56	y = (2460+x)*x = 2464*4 = 9856	
00 00	Algorithm terminates: Answer is 12.34	

Example (3):

Find the square root of 2 .

Solution:

√ 02.00 00 00 00 is

1. 4 1 4 2

02	1*1 <= 2 < 2*2	x = 1
01	y = x*x = 1*1 = 1	
01 00	24*4 <= 100 < 25*5	x = 4
00 96	y = (20+x)*x = 24*4 = 96	

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04 00   281*1 <= 400 < 282*2   x = 1
02 81   y = (280+x)*x = 281*1 = 281
01 19 00   2824*4 <= 11900 < 2825*5   x = 4
01 12 96   y = (2820+x)*x = 2824*4 = 11296
06 04 00   28282*2 <= 60400 < 28283*3   x = 2

The desired precision is achieved:
The square root of 2 is about 1.4142

Example (4) by discussion:
Consider the perfect square 2809 = 53^2. Use the duplex method to find the square root of 2,809.

Solution:

- Set down the number in groups of two digits.
- Define a divisor, a dividend and a quotient to find the root.
- Given 2809. Consider the first group, 28.
    - Find the nearest perfect square below that group.
    - The root of that perfect square is the first digit of our root.
    - Since 28 > 25 and 25 = 5^2, take 5 as the first digit in the square root.
    - For the divisor take double this first digit (2 · 5), which is 10.
- Next, set up a division framework with a colon.
    - 28: 0 9 is the dividend and 5: is the quotient.
    - Put a colon to the right of 28 and 5 and keep the colons lined up vertically. The duplex is calculated only on quotient digits to the right of the colon.
- Calculate the remainder. 28: minus 25: is 3:
    - Append the remainder on the left of the next digit to get the new dividend.
    - Here, append 3 to the next dividend digit 0, which makes the new dividend 30. The divisor 10 goes into 30 just 3 times. (No reserve needed here for subsequent deductions.)
- Repeat the operation.
    - The zero remainder appended to 9. Nine is the next dividend.
    - This provides a digit to the right of the colon so deduct the duplex, 3^2 = 9.
    - Subtracting this duplex from the dividend 9, a zero remainder results.
    - Ten into zero is zero. The next root digit is zero. The next duplex is 2(3·0) = 0.
    - The dividend is zero. This is an exact square root, 53.

Example (5): (analysis and square root framework):
Find the square root of 2809.

Solution :
Set down the number in groups of two digits. The number of groups gives the number of whole digits in the root. Put a colon after the first group, 28, to separate it. From the first group, 28, obtain the divisor, 10, since 28 > 25=5^2 and by doubling this first root, 2x5=10.

Gross dividend: 28: 0 9. Using mental math:
Divisor: 10) 3 0 Square: 10) 28: $_3$0 9
Duplex, Deduction: 25: xx 09 Square root: 5: 3. 0
Dividend: 30 00
Remainder: 3: 0 0 0 0
Square Root, Quotient: 5: 3. 0

Example (6) :
Find the square root of 2.080180881

Solution:
By the duplex method: this ten-digit square has five digit-pairs, so it will have a five-digit square root. The first digit-pair is 20. Put the colon to the right. The nearest square below 20 is 16, whose root is 4, the first root digit. So, use 2·4=8 for the divisor. Now proceed with the duplex division, one digit column at a time. Prefix the remainder to the next dividend digit.

divisor; gross dividend: 8) 20: 8 0 1 8 0 8 8 1
read the dividend diagonally up: 4 8 7 11 10 10 0 8
minus the duplex: 16: xx 25 60 36 90 108 00 81
actual dividend: : 48 55 11 82 10 00 08 00
minus the product: : 40 48 00 72 00 00 0 00
remainder: 4: 8 7 11 10 10 0 8 00
quotient: 4: 5, 6 0 9. 0 0 0 0

Duplex calculations:
Quotient-digits ==> Duplex deduction.
5 ==> 5^2= 25
5 and 6 ==> 2(5·6) = 60
5,6,0 ==> 2(5·0)+6^2 = 36
5,6,0,9 ==> 2(5·9)+2(6·0) = 90
5,6,0,9,0 ==> 2(5·0)+2(6·9)+ 0 = 108
5,6,0,9,0,0 ==> 2(5·0)+2(6·0)+2(0·9) = 0
5,6,0,9,0,0,0 ==> 2(5·0)+2(6·0)+2(0·0)+9^2 = 81
Hence the square root of 2,080,180,881 is exactly 45,609.

Example (7):
Find the square root of two to ten places.

Solution:
Take 20,000 as the beginning group, using three digit-pairs at the start. The perfect square just below 20,000 is 141, since 141^2 = 19881 < 20,000. So, the first root digits are 141 and the divisor doubled, 2 x 141 = 282. With a larger divisor the duplex will be relatively small. Hence, the multiple of the divisor can be picked without confusion.

Dividend: 2.0000 : 0 0 0 0 0 0 0 0
Diagonal ;Divisor: (282) : 1190 620 400 1020 1620 1820 750 1120
Minus duplex: : xxxx 16 16 12 28 53 74 59
Actual dividend: 20000 : 1190 604 384 1008 1592 1767 676 1061
Minus product: 19881 : 1128 564 282 846 1410 1692 564 846
Remainder: 119 : 62 40 102 162 182 75 112 215
Root quotient: 1.41 : 4 2 1 3 5 6 2 3

Ten multiples of 282: 282; 564; 846; 1128; 1410; 1692; 1974; 2256; 2538; 2820.

IV. ITERATIVE METHODS FOR RECIPROCAL SQUARE ROOTS

The following are iterative methods for finding the reciprocal square root of *S* which is $1/\sqrt{S}$. Once it has been found, find $\sqrt{S}$ by simple multiplication: $\sqrt{S} = S \cdot (1/\sqrt{S})$. These iterations involve only multiplication, and not division. They are therefore faster than the Babylonian method. However, they are not stable. If the initial value is not close to the reciprocal square root, the iterations will diverge away from it rather than converge to it. It can therefore be advantageous to perform an iteration of the Babylonian method on a rough estimate before

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starting to apply these methods.

- One method is found by applying Newton's method to the equation $(1 / x^2) - S = 0$. It converges quadratically:

$$x_{n+1} = \frac{x_n}{2} \cdot (3 - S \cdot x_n^2).$$

- Another iteration obtained by Halley's method, which is the Householder's method of order two, converges cubically, but involves more operations per iteration:

$$y_n = S \cdot x_n^2 ,$$

$$x_{n+1} = \frac{x_n}{8} \cdot (15 - y_n \cdot (10 - 3 \cdot y_n)).$$

VI. TAYLOR SERIES

If N is an approximation to $\sqrt{S}$, a better approximation can be found by using the Taylor series of the square root function:

$$\sqrt{N^2+d} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! d^n}{(1-2n) n!^2 4^n N^{2n-1}} = N + \frac{}{2}$$

As an iterative method, the order of convergence is equal to the number of terms used. With 2 terms, it is identical to the Babylonian method; With 3 terms, each iteration takes almost as many operations as the Bakhshali approximation, but converges more slowly. Therefore, this is not a particularly efficient way of calculation.

V. CONTINUED FRACTION EXPANSION

Quadratic irrationals (numbers of the form $\frac{a+\sqrt{b}}{c}$, where a, b and c are integers), and in particular, square roots of integers, have periodic continued fractions. Sometimes what is desired is finding not the numerical value of a square root, but rather its continued fraction expansion. The following iterative algorithm can be used for this purpose (S is any natural number that is not a perfect square):

$$m_0 = 0$$
$$d_0 = 1$$
$$a_0 = \left\lfloor \sqrt{S} \right\rfloor$$
$$m_{n+1} = d_n a_n - m_n$$
$$d_{n+1} = \frac{S - m_{n+1}^2}{d_n}$$
$$a_{n+1} = \left\lfloor \frac{\sqrt{S} + m_{n+1}}{d_{n+1}} \right\rfloor = \left\lfloor \frac{a_0 + m_{n+1}}{d_{n+1}} \right\rfloor.$$

Notice that m_n, d_n, and a_n are always integers. The algorithm terminates when this triplet is the same as one encountered before. The expansion will repeat from then on. The sequence $[a_0; a_1, a_2, a_3, \ldots]$ is the continued fraction expansion:

$$\sqrt{S} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Example (7):

Find the square root of 114 as a continued fraction

Solution:

Begin with $m_0 = 0$; $d_0 = 1$; and $a_0 = 10$ ($10^2 = 100$ and $11^2 = 121 > 114$ so 10 chosen).

$$\sqrt{114} = \frac{\sqrt{114}+0}{1} = 10 + \frac{\sqrt{114}-10}{1} = 10 + \frac{(\sqrt{114}-10)(\sqrt{114}+10)}{\sqrt{114}+10}$$

$$= 10 + \frac{114-100}{\sqrt{114}+10} = 10 + \frac{1}{\frac{\sqrt{114}+10}{14}}.$$

$$m_1 = d_0 \cdot a_0 - m_0 = 1 \cdot 10 - 0 = 10 .$$

$$d_1 = \frac{S - m_1^2}{d_0} = \frac{114 - 10^2}{1} = 14 .$$

$$a_1 = \left\lfloor \frac{a_0 + m_1}{d_1} \right\rfloor = \left\lfloor \frac{10+10}{14} \right\rfloor = \left\lfloor \frac{20}{14} \right\rfloor = 1 .$$

So, $m_1 = 10$; $d_1 = 14$; and $a_1 = 1$.

$$\frac{\sqrt{114}+10}{14} = 1 + \frac{\sqrt{114}-4}{14} = 1 + \frac{114-16}{14(\sqrt{114}+4)} = 1 + \frac{1}{\frac{\sqrt{114}+4}{7}}.$$

Next, $m_2 = 4$; $d_2 = 7$; and $a_2 = 2$.

$$\frac{\sqrt{114}+4}{7} = 2 + \frac{\sqrt{114}-10}{7} = 2 + \frac{14}{7(\sqrt{114}+10)} = 2 + \frac{1}{\frac{\sqrt{114}+10}{2}}.$$

$$\frac{\sqrt{114}+10}{2} = 10 + \frac{\sqrt{114}-10}{2} = 10 + \frac{14}{2(\sqrt{114}+10)} = 10 + \frac{1}{\frac{\sqrt{114}+10}{7}}.$$

$$\frac{\sqrt{114}+10}{7} = 2 + \frac{\sqrt{114}-4}{7} = 2 + \frac{98}{7(\sqrt{114}+4)} = 2 + \frac{1}{\frac{\sqrt{114}+4}{14}}.$$

$$\frac{\sqrt{114}+4}{14} = 1 + \frac{\sqrt{114}-10}{14} = 1 + \frac{14}{14(\sqrt{114}+10)} = 1 + \frac{1}{\frac{\sqrt{114}+10}{1}}.$$

$$\frac{\sqrt{114}+10}{1} = 20 + \frac{\sqrt{114}-10}{1} = 20 + \frac{14}{\sqrt{114}+10} = 20 + \frac{1}{\frac{\sqrt{114}+10}{14}}.$$

Now, loop back to the second equation above. Consequently, the simple continued fraction for the square root of 114 is

$$\sqrt{114} = [10; 1, 2, 10, 2, 1, 20, 1, 2, 10, 2, 1, 20, 1, 2, 10, 2, 1, 20, \ldots].$$

Its actual value is approximately 10.67707 82520 31311 21....

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Generalized continued fraction:
A more rapid method is to evaluate its generalized continued fraction. From the formula derived there:

$$\sqrt{z}=\sqrt{x^2+y}=x+\cfrac{y}{2x+\cfrac{y}{2x+\cfrac{y}{2x+\ddots}}}=x+\cfrac{2x\cdot y}{2(2z-y)-y-\cfrac{y^2}{2(2z-y)-\cfrac{y^2}{2(2z-y)-\ddots}}}$$

the square root of 114 is quickly found:

$$\sqrt{114}=\sqrt{11^2-7}=11-\cfrac{7}{22-\cfrac{7}{22-\cfrac{7}{22-\ddots}}}=11-\cfrac{22\cdot 7}{470+7-\cfrac{49}{470-\cfrac{49}{470-\ddots}}}.$$

Furthermore, the fact that 114 is 2/3 of the way between $10^2=100$ and $11^2=121$ results in

$$\sqrt{114}=\frac{\sqrt{1026}}{3}=\frac{\sqrt{32^2+2}}{3}=\frac{32}{3}+\cfrac{2/3}{64+\cfrac{2}{64+\cfrac{2}{64+\cfrac{2}{64+\ddots}}}},$$

which is simply the aforementioned [10;1,2, 10,2,1, 20,1,2, 10,2,1, 20,1,2, ...] evaluated at every third term. Combining pairs of fractions produces

$$\sqrt{114}=\frac{\sqrt{32^2+2}}{3}=\frac{32}{3}+\cfrac{64/3}{2050-1-\cfrac{1}{2050-\cfrac{1}{2050-\cfrac{1}{2050-\ddots}}}},$$

which is now [10;1,2, 10,2,1,20,1,2, 10,2,1,20,1,2, ...] evaluated at the third term and every six terms thereafter.

Section(3)

VII. NEWTON'S METHOD

Introduction:

In this section we are going to look at a method for approximating solutions to equations. We all know that equations need to be solved on occasion and in fact we've solved quite a few equations ourselves to this point. In all the examples we've looked at to this point we were able to actually find the solutions, but it's not always possible to do that exactly and/or do the work by hand. That is where this application comes into play. So, let's see what this application is all about. Newton's Method attempts to construct a sequence x_n from an initial guess x_0 that converges towards x_* such that $f'(x_*)=0$. This is x_* called a stationary point of f.

The second order Taylor expansion $f_T(x)$ of the function around x_n (where $\Delta x = x - x_n$) is:

$$f_T(x_n+\Delta x)=f_T(x)=f(x_n)+f'(x_n)\Delta x+\frac{1}{2}f''(x_n)\Delta x^2$$

, attains its extremum when its derivative with respect to Δx is equal to zero, i.e. when Δx solves the linear equation:

$$f'(x_n)+f''(x_n)\Delta x=0.$$

(Considering the right-hand side of the above equation as a quadratic in Δx, with constant coefficients.) Thus, provided that $f(x)$ is a twice-differentiable function well approximated by its second order Taylor expansion and the initial guess x_0 is chosen close enough to x_*, the sequence

$$\Delta x = x - x_n = -\frac{f'(x_n)}{f''(x_n)}$$

(x_n) defined by:

$$x_{n+1}=x_n-\frac{f'(x_n)}{f''(x_n)},\ n=0,1,\ldots$$

will converge towards a root of f', i.e. x_* for which $f'(x_*)=0$.

Newton's method :

Newton's method (also known as the Newton–Raphson method), named after Isaac Newton and Joseph Raphson, is a method for finding successively better approximations to the roots (or zeroes) of a real-valued function. The algorithm is first in the class of Householder's methods, succeeded by Halley's method.

VIII. THE NEWTON-RAPHSON METHOD IN ONE VARIABLE

Given a function $f(x)$ and its derivative $f'(x)$, we begin with a first guess x_0 for a root of the function. Provided the function is reasonably well-behaved a better approximation x_1 is

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}.$$

Geometrically, x_1 is the intersection with the x-axis of a line tangent to f at $f(x_0)$. The process is repeated until a sufficiently accurate value is reached:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

The idea of the method is as follows: one starts with an initial guess which is reasonably close to the true root, then the function is approximated by its tangent line (which can be computed using the tools of calculus), and one computes the x-intercept of this tangent line (which is easily done with elementary algebra). This x-intercept will typically be a better approximation to the function's root than the original guess, and the method can be iterated. Suppose $f : [a, b] \to \mathrm{R}$ is a differentiable function defined on the interval $[a, b]$ with values in the real numbers R. The formula for converging on the root can be easily derived. Suppose we have some current approximation x_n. Then we can derive the formula for a better approximation, x_{n+1} by

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referring to the diagram on the right. We know from the definition of the derivative at a given point that it is the slope of a tangent at that point. That is

$$f'(x_n) = \frac{\Delta y}{\Delta x} = \frac{f(x_n) - 0}{x_n - x_{n+1}}.$$

Here, f' denotes the derivative of the function f. Then by simple algebra we can derive

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We start the process off with some arbitrary initial value x_0. (The closer to the zero, the better. But, in the absence of any intuition about where the zero might lie, a "guess and check" method might narrow the possibilities to a reasonably small interval by appealing to the intermediate value theorem.) The method will usually converge, provided this initial guess is close enough to the unknown zero, and that $f'(x_0) \neq 0$. Furthermore, for a zero of multiplicity 1, the convergence is at least quadratic (see rate of convergence) in a neighborhood of the zero, which intuitively means that the number of correct digits roughly at least doubles in every step. More details can be found in the analysis section below. The Householder's methods are similar but have higher order for even faster convergence. However, the extra computations required for each step can slow down the overall performance relative to Newton's method, particularly if f or its derivatives are computationally expensive to evaluate.

IX. PROOF OF QUADRATIC CONVERGENCE FOR NEWTON'S ITERATIVE METHOD

According to Taylor's theorem, any function $f(x)$ which has a continuous second derivative can be by an expansion about a point that is close to a root of $f(x)$. Suppose this root is α .Then the expansion of $f(\alpha)$ about x_n is:

$$f(\alpha) = f(x_n) + f'(x_n)(\alpha - \quad (1)$$

where the Lagrange form of the Taylor series expansion remainder is

$$R_1 = \frac{1}{2!} f''(\xi_n)(\alpha - x_n)^2,$$

where ξ_n is in between x_n and α .

Since α is the root, (1) becomes:

$$0 = f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + \frac{1}{2} f''(\xi_n)(\alpha - x_n)^2 \quad (2)$$

Dividing equation (2) by $f'(x_n)$ and rearranging gives

$$\frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) = \frac{-f''(\xi_n)}{2f'(x_n)}(\alpha \cdot \quad (3)$$

Remembering that x_{n+1} is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (4)$$

one finds that

$$\underbrace{\alpha - x_{n+1}}_{\epsilon_{n+1}} = \frac{-f''(\xi_n)}{2f'(x_n)} (\underbrace{\alpha - x_n}_{\epsilon_n})^2 .$$

That is,

$$\epsilon_{n+1} = \frac{-f''(\xi_n)}{2f'(x_n)} {\epsilon_n}^2 . \quad (5)$$

Taking absolute value of both sides gives

$$|\epsilon_{n+1}| = \frac{|f''(\xi_n)|}{2\,|f'(x_n)|} {\epsilon_n}^2 \quad (6)$$

Equation (6) shows that the rate of convergence is quadratic if 1-

$f'(x) \neq 0; \forall x \in I$, where I is the interval $[\alpha - r, \alpha + r]$ for some $r \geq |(\alpha - x_0)|$;

following conditions are satisfied:

1. $f''(x)$ is finite $, \forall x \in I;$
2. x_0 sufficiently close to the root α

The term sufficiently close in this context means the following:

(a) Taylor approximation is accurate enough such that we can ignore higher order terms,

(b)

$$\frac{1}{2}\left|\frac{f''(x_n)}{f'(x_n)}\right| < C \left|\frac{f''(\alpha)}{f'(\alpha)}\right|, \text{ for some } C < \infty,$$

(c)

$$C\left|\frac{f''(\alpha)}{f'(\alpha)}\right|\epsilon_n < 1, \text{ for } n \in \mathbb{Z}^+ \cup \{0\} \text{ and } C \text{ satisfying condition (b)}.$$

Finally, (6) can be expressed in the following way:

$$|\epsilon_{n+1}| \leq M \epsilon^2{}_n$$

where M is the supremum of the variable coefficient of $\epsilon^2{}_n$ on the interval I defined in the condition 1, that is:

$$M = \sup_{x \in I} \frac{1}{2}\left|\frac{f''(x)}{f'(x)}\right|.$$

The initial point x_0 has to be chosen such that conditions (1) through (3) are satisfied, where the third condition requires that $M\,|\epsilon_0| < 1.$

X. MINIMIZATION AND MAXIMIZATION PROBLEMS (NEWTON'S METHOD IN OPTIMIZATION)

Newton's method can be used to find a minimum or maximum of a function. The derivative is zero at a minimum or maximum, so minima and maxima can be found by applying Newton's method to the derivative. The iteration becomes:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}.$$

Examples: (Square root of a number)

Consider the problem of finding the square root of a number. There are many methods of computing square roots, and Newton's method is one.

Example(8):

Find the square root of 612, this is equivalent to finding the solution to $x^2 = 612$

The function to use in Newton's method is then,

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$$f(x) = x^2 - 612$$

Where $f'(x) = 2x$

With an initial guess of 10, the sequence given by Newton's method is:

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} &=& \quad 10 - \frac{10^2 - 612}{2 \cdot 10} &=& \; 35.6 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} &=& \; 35.6 - \frac{35.6^2 - 612}{2 \cdot 35.6} &=& \; \underline{2}6.3955056 \\
x_3 &= \quad \vdots &=& \quad \vdots &=& \; \underline{24.7}906355 \\
x_4 &= \quad \vdots &=& \quad \vdots &=& \; \underline{24.73868}83 \\
x_5 &= \quad \vdots &=& \quad \vdots &=& \; \underline{24.7386338}
\end{aligned}$$

Where the correct digits are underlined. With only a few iterations one can obtain a solution accurate to many decimal places.

Example(9):

Solve : $\cos(x) = x^3$

Solution:

Consider the problem of finding the positive number x with $\cos(x) = x^3$. We can rephrase that as finding the zero of $f(x) = \cos(x) - x^3$. We have $f'(x) = -\sin(x) - 3x^2$. Since $\cos(x) \le 1$ for all x and $x^3 > 1$ for $x > 1$, we know that our zero lies between 0 and 1. We try a starting value of $x_0 = 0.5$. (Note that a starting value of 0 will lead to an undefined result, showing the importance of using a starting point that is close to the zero.)

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} &=& \; 0.5 - \frac{\cos(0.5) - (0.5)^3}{-\sin(0.5) - 3(0.5)^2} &=& \; 1.112141637097 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} &=& \quad \vdots &=& \; \underline{0.}909672693736 \\
x_3 &= \quad \vdots &=& \quad \vdots &=& \; \underline{0.86}7263818209 \\
x_4 &= \quad \vdots &=& \quad \vdots &=& \; \underline{0.86547}7135298 \\
x_5 &= \quad \vdots &=& \quad \vdots &=& \; \underline{0.8654740331}11 \\
x_6 &= \quad \vdots &=& \quad \vdots &=& \; \underline{0.865474033102}
\end{aligned}$$

The correct digits are underlined in the above example. In particular, x_6 is correct to the number of decimal places given. We see that the number of correct digits after the decimal point increases from 2 (for x_3) to 5 and 10, illustrating the quadratic convergence. Sometimes we are presented with a problem which cannot be solved by simple algebraic means. For instance, if we needed to find the roots of the polynomial $x^3 - x + 1 = 0$, we would find that the tried and true techniques just wouldn't work. However, we will see that calculus gives us a way of finding approximate solutions.

A Simple Example :

Let's start by computing $\sqrt{5}$. Of course, this is easy if you have a calculator, but it is a simple example which will illustrate a more general method. First, we'll think about the problem in a slightly different way. We are looking for $\sqrt{5}$ which is a solution of the equation $f(x) = x^2 - 5 = 0$. The problem is that it is difficult to generate a numerical solution to this equation. But remember in the section on approximations , we saw how to approximate a function near a given point by its tangent line. The idea here will be to actually solve the approximate equation which is easy since it is a linear one.

If we think for a minute, we know that $\sqrt{5}$ is between 2 and 3 so let's just choose to use the linear approximation at $x_0 = 2$. We know that $f(x) = x^2 - 5$ so that $f'(x) = 2x$. The linear approximation is then

$$\begin{aligned}
f(x) \approx l(x) &= f(x_0) + f'(x_0)(x - x_0) \\
f(x) \approx l(x) &= -1 + 4(x - 2) = 4x - 9
\end{aligned}$$

Notice that the linear equation is easy to solve. We will then approximate the solution to $f(x) = 0$ by the solution to $l(x) = 4x - 9 = 0$ which is $x = \frac{9}{4} = 2.25$. If you have a look on a calculator, you will see that $\sqrt{5} = 2.236068\ldots$. So you can see that we have found a fairly good approximation.

We can understand what we have done graphically. We are looking for a solution to $f(x) = 0$ which is where the graph of $f(x)$ crosses the X-axis. We approximate that point by the point where the tangent line crosses the X-axis

Now this is where the story becomes interesting since we can repeat what we have just done using the new approximate solution. That is, we will call $x_1 = \frac{9}{4}$ and consider the linear approximation at that point.

$$\begin{aligned}
f(x) \approx l(x) &= f(x_1) + f'(x_1)(x - x_1) \\
f(x) \approx l(x) &= (\frac{81}{16} - 5) + \frac{9}{2}(x - \frac{9}{4}) = \frac{9}{2}x - \frac{161}{16}
\end{aligned}$$

Now if we call x_2 the solution to $l(x) = 0$, we find that $x_2 = 2.236111$ which is an even better approximate solution to the equation. We could continue this process generating better approximations to $\sqrt{5}$ at every step. This is the basic idea of a technique known as *Newton's Method*

The general method:

More generally, we can try to generate approximate solutions to the equation $f(x) = 0$ using the same idea. Suppose that x_0 is some point which we suspect is near a solution. We can form the linear approximation at x_0 and solve the linear equation instead.

That is, we will call x_1 the solution to $l(x) = f(x_0) + f'(x_0)(x - x_0) = 0$. In other words,

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$$f(x_0) + f'(x_0)(x_1 - x_0) = 0$$
$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If our first guess x_0 was a good one, the approximate solution x_1 should be an even better approximation to the solution of $f(x) = 0$. Once we have x_1, we can repeat the process to obtain x_2, the solution to the linear equation

$$l(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) = 0$$

Solving in the same way, we see that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Maybe now you see that we can repeat this process indefinitely: from x_2, we generate x_3 and so on. If, after n steps, we have an approximate solution x_n, then the next step is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Provided we have started with a good value for x_0, this will produce approximate solutions to any degree of accuracy.

XI. PRACTICAL CONSIDERATIONS

Newton's method is an extremely powerful technique—in general the convergence is quadratic: as the method converges on the root, the difference between the root and the approximation is squared (the number of accurate digits roughly doubles) at each step. However, there are some difficulties with the method shown at the end of this thiese in the appendix.

Section(4)

A Matlab program to the Newton's method applied to some problems:

Example (1):

This example solves the system of two equations and two unknowns:

$$2x_1 - x_2 = e^{-x_1}$$
$$-x_1 + 2x_2 = e^{-x_2}.$$

Start your search for a solution at x_0 = [-5 -5].

```
x =
   0.5671
   0.5671
fval =
  1.0e-006 *
    -0.4059
    -0.4059
```

Example (2):

Find a matrix X that satisfies the equation

$$X * X * X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

starting at the point X= [1,1; 1,1].

```
x =
   -0.1291    0.8602
    1.2903    1.1612

Fval =
  1.0e-009 *
   -0.1621    0.0780
    0.1164   -0.0467

exitflag =
    1
ans =
  4.8081e-020
```

Notes:

If the system of equations is linear, use\ (matrix left division) for better speed and accuracy. For example, to find the solution to the following linear system of equations:

$3x_1 + 11x_2 - 2x_3 = 7$
$x_1 + x_2 - 2x_3 = 4$
$x_1 - x_2 + x_3 = 19.$

Formulate and solve the problem as

```
A = [ 3 11 -2; 1 1 -2; 1 -1 1];
b = [ 7; 4; 19];
x = A\b
x =
   13.2188
   -2.3438
    3.4375
```

XII. APPENDIX

Difficulty in calculating derivative of a function:

Newton's method requires that the derivative be calculated directly. An analytical expression for the derivative may not be easily obtainable and could be expensive to evaluate. In these situations, it may be appropriate to approximate the derivative by using the slope of a line through two nearby points on the function. Using this approximation would result in something like the secant method whose convergence is slower than that of Newton's method. Failure of the method to converge to the root:

It is important to review the proof of quadratic convergence of Newton's Method [2] before implementing it. Specifically, one should review the assumptions made in the proof. For situations where the method fails to converge, it is because the assumptions made in this proof are not met.

Overshoot:

If the first derivative is not well behaved in the neighborhood of the root, the method may overshoot, and diverge from the desired root. Furthermore, if a stationary point of the function is encountered, the derivative is zero and the method will terminate due to division by zero.

Poor initial estimate:

A large error in the initial estimate can contribute to non-convergence of the algorithm. Mitigation of non-convergence:

In a robust implementation of Newton's method, it is common to place limits on the number of iterations, bound the solution to an interval known to contain the root, and combine the method with a more robust root finding method.

Slow convergence for roots of multiplicity > 1:

If the root being sought has multiplicity greater than one, the

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convergence rate is merely linear (errors reduced by a constant factor at each step) unless special steps are taken. When there are two or more roots that are close together then it may take many iterations before the iterates get close enough to one of them for the quadratic convergence to be apparent. However, if the multiplicity *m* of the root is known, one can use the following modified algorithm that preserves the quadratic convergence rate:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}.$$

XIII. ANALYSIS

Suppose that the function f has a zero at α, i.e., $f(\alpha) = 0$.

If f is continuously differentiable and its derivative is nonzero at α, then there exists a neighborhood of α such that for all starting values x_0 in that neighborhood, the sequence $\{x_n\}$ will converge to α. If the function is continuously differentiable and its derivative is not 0 at α and it has a second derivative at α then the convergence is quadratic or faster. If the second derivative is not 0 at α then the convergence is merely quadratic [5].If the third derivative exists and is bounded in a neighborhood of α, then:

$$\Delta x_{i+1} = \frac{f''(\alpha)}{2f'(\alpha)}(\Delta x_i)^2 + O[\Delta x_i]^3\,,$$

where $\Delta x_i \triangleq x_i - \alpha\,.$

If the derivative is 0 at α, then the convergence is usually only linear. Specifically, if f is twice continuously differentiable, $f'(\alpha) = 0$ and $f''(\alpha) \neq 0$, then there exists a neighborhood of α such that for all starting values x_0 in that neighborhood, the sequence of iterates converges linearly, with rate $\log_{10} 2$ (Süli & Mayers,). Alternatively if $f'(\alpha) = 0$ and $f'(x) \neq 0$ for $x \neq 0$, x in a neighborhood U of α, α being a zero of multiplicity r, and if $f \in C^r(U)$ then there exists a neighborhood of α such that for all starting values x_0 in that neighborhood, the sequence of iterates converges linearly. However, even linear convergence is not guaranteed in pathological situations. In practice these results are local and the neighborhood of convergence are not known a priori, but there are also some results on global convergence, for instance, given a right neighborhood U_+ of α, if f is twice differentiable in U_+ and if $f' \neq 0$, $f \cdot f'' > 0$ in U_+, then, for each x_0 in U_+ the sequence x_k is monotonically decreasing to α.

XIV. FAILURE ANALYSIS

Newton's method is only guaranteed to converge if certain conditions are satisfied. If the assumptions made in the proof of Quadratic Convergence are met, the method will converge. For the following subsections, failure of the method to converge indicates that the assumptions made in the proof were not met.

XV. BAD STARTING POINTS

In some cases the conditions on function necessary for convergence are satisfied, but the point chosen as the initial point is not in the interval where the method converges. In such cases a different method, such as bisection, should be used to obtain a better estimate for the zero to use as an initial point.

Iteration point is stationary:

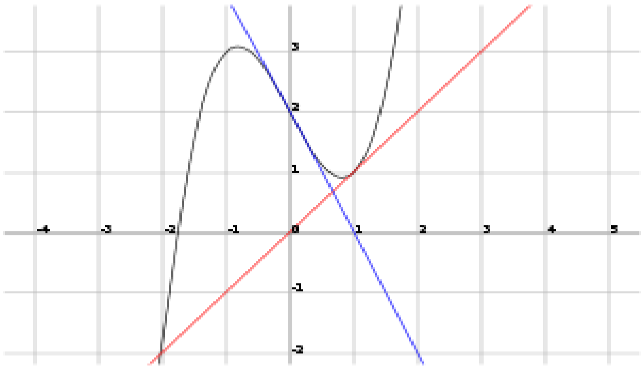
Consider the function:

$$f(x) = 1 - x^2.$$

It has a maximum at x=0 and solutions of $f(x) = 0$ at $x = \pm 1$. If we start iterating from the stationary point x_0=0 (where the derivative is zero), x_1 will be undefined, since the tangent at (0,1) is parallel to the x-axis:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{0}.$$

The same issue occurs if, instead of the starting point, any iteration point is stationary. Even if the derivative is small but not zero, the next iteration will be a far worse approximation Starting point enters a cycle:



The tangent lines of $x^3 - 2x + 2$ at 0 and 1 intersect the x-axis at 1 and 0 respectively, illustrating why Newton's method oscillates between these values for some starting points[7].

For some functions, some starting points may enter an infinite cycle, preventing convergence. Let

$$f(x) = x^3 - 2x + 2$$

and take 0 as the starting point. The first iteration produces 1 and the second iteration returns to 0 so the sequence will alternate between the two without converging to a root. In general, the behavior of the sequence can be very complex. (See Newton fractal.)

XVI. DISCONTINUOUS DERIVATIVE

If the derivative is not continuous at the root, then convergence may fail to occur in any neighborhood of the root. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x + x^2 \sin\left(\frac{2}{x}\right) & \text{if } x \neq 0. \end{cases}$$

Its derivative is:

$$f'(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1 + 2x\sin\left(\frac{2}{x}\right) - 2\cos\left(\frac{2}{x}\right) & \text{if } x \neq 0. \end{cases}$$

Within any neighborhood of the root, this derivative keeps changing sign as x approaches 0 from the right (or from the left) while $f(x) \geq x - x^2 > 0$ for $0 < x < 1$.

So $f(x)/f'(x)$ is unbounded near the root, and Newton's method will diverge almost everywhere in any neighborhood of it, even though:

- the function is differentiable (and thus continuous)

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everywhere;

- the derivative at the root is nonzero;
- *f* is infinitely differentiable except at the root; and
- the derivative is bounded in a neighborhood of the
- root (unlike $f(x)/f'(x)$).

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